ADVANCED MATHEMATICS PAPER 1 (a)

PRE- NATIONAL EXAMINATION (CSSC) FORM SIX 2022/2023

MARKING SCHEME

1. a) 0.3114
   b) 0.3082
   c) 74.6900 minutes
   d) 52.4656

2.a) i) $y = coshx - 3sinhx$ meets $y = -1$ at $(k, -1)$

This gives: $-1 = coshk - 3sinhk$

$-1 = \frac{e^k + e^{-k}}{2} - 3\left(\frac{e^x - e^{-x}}{2}\right)$

$-2 = -2e^k + 4e^{-k}$

$-1 = -e^k + 2e^{-k}$

$-e^k = -e^{2k} + 2$

$e^{2k} - e^k - 2 = 0$

Therefore, $\boldsymbol{e^{2k} - e^k - 2 = 0}$

Hence shown!

ii)From: $e^{2k} - e^k - 2 = 0$

$(e^k - 2)(e^k + 1) = 0$

$e^k - 2 = 0$

$(e^k = 2$

$k = ln2$

Therefore, $\boldsymbol{k = ln2}$

b) Given $6cosh\frac{1}{6}x - sinh3x$

Consider the power series expansion for $cosh\frac{1}{6}x$

Let $y = \frac{x}{6}$

$\cosh y = 1 + \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \frac{1}{6!}y^6 + \cdots$

This means,

$$cosh\frac{1}{6}x = 1 + \frac{1}{2!}\left(\frac{x}{6}\right)^2 + \frac{1}{4!}\left(\frac{x}{6}\right)^4 + \frac{1}{6!}\left(\frac{x}{6}\right)^6 + \cdots$$

$$= 1 + \frac{x^2}{72} + \frac{x^4}{31104} + \frac{x^6}{33592320} + \cdots.$$

$$6cosh\frac{1}{6}x = 6 + \frac{x^2}{12} + \frac{x^4}{5184} + \frac{x^6}{559820} + \cdots$$

Also, for the power series expansion for $sinh3x$

Let $y = 3x$

From: $sinhy = y + \frac{1}{3!}y^3 + \frac{1}{5!}y^5 + \cdots$

$$sinh3x = 3x + \frac{1}{3!}(3x)^3 + \frac{1}{5!}(3x)^5 + \cdots$$

$$= 3x + \frac{9}{2}x^3 + \frac{81}{40}x^5 + \cdots$$

Thus,

$$6cosh\frac{1}{6}x - sinh3x = \left(6 + \frac{x^2}{12} + \frac{x^4}{5184} + \frac{x^6}{559820} + \cdots\right) - \left(3x + \frac{9}{2}x^3 + \frac{81}{40}x^5 + \cdots\right)$$

Therefore,

$$\boldsymbol{6cosh\frac{1}{6}x - sinh3x = 6 - 3x + \frac{x^2}{12} - \frac{9}{2}x^3 + \frac{x^4}{5184} - \frac{81}{40}x^5 + \frac{x^6}{559820}}.$$

c) Given $y = 80cosh\left(\frac{x}{80}\right)$.

$$\frac{dy}{dx} = sinh\left(\frac{x}{80}\right)$$

At minimum value, $\frac{dy}{dx} = 0$

$$sinh\left(\frac{x}{80}\right) = 0$$

$$e^{\frac{x}{40}} - 1 = 0$$

$$\frac{x}{40} = \ln(1)$$

$$x = 40\ln(1)$$

$$x = 40(0) = 0$$

At minimum value, $x = 0$

But $y = 80cosh\left(\frac{x}{80}\right)$.

Substituting $x = 0$,

$y = 80cosh(0)$.

$y = 80(1) = 80$

Therefore, the minimum value of telegraph wire is **80.**

3. (a)Let x be number of drives through restaurants and y be number of full service restaurants

Objective function

To maximize f(x,y) = 200,000x+500,000y

Constraints

100,000x +150,000y ≤ 2400,000 i.e. 2x+3y ≤ 48

5x+15y ≤ 210 i.e. x+3y ≤ 42

x+y ≤ 20

Non- negative constraints

x ≥ 0, y ≥ 0

(b)Let x be number of packets of medicines from A to P

Let y be number of packets of medicines from A to Q

Let z be number of packets of medicines from A to R

$$x + y + z = 6000$$

$$z = 6000 - x - y$$

Objective function

$$Transport\ cost = 500x + 400y + 300z + 400(4000 - x) + 200(4000 - y) + 500(5000 - z)$$

$$= 500x + 400y + 300z + 1600000 - 400x + 800000 - 200y + 2500000 - 500z$$

$$= 100x + 200y - 200z + 4900000$$

$$= 100x + 200y - 200(6000 - x - y) + 4900000$$

$$= 100x + 200y - 1200000 + 200x + 200y + 4900000$$

$$Transport\ cost\ = 300x + 400y + 3{,}700{,}000$$

Constraints;

$x \leq 4000.$

y≤4000

$z \leq 5000$

$6000 - x - y \leq 5000$

$x + y \geq 1000.$

$x \geq 0.$

$y \geq 0.$

$z \geq 0$

$6000 - x - y \geq 0$

$x + y \leq 6000.$

Corner Point	Transport Cost
A(0,4000)	5,300,000
B(2000,4000)	5,900,000
C(4000,2000)	5,700,000
D(4000,0)	4,900,000
E(1000,0)	4,000,000
F(0,1000)	4,100,000

Optimum point is (1000, 0)

$z = 6000 - 1000 - 0 = 5000$

The company should supply

1,000 packets from A to P

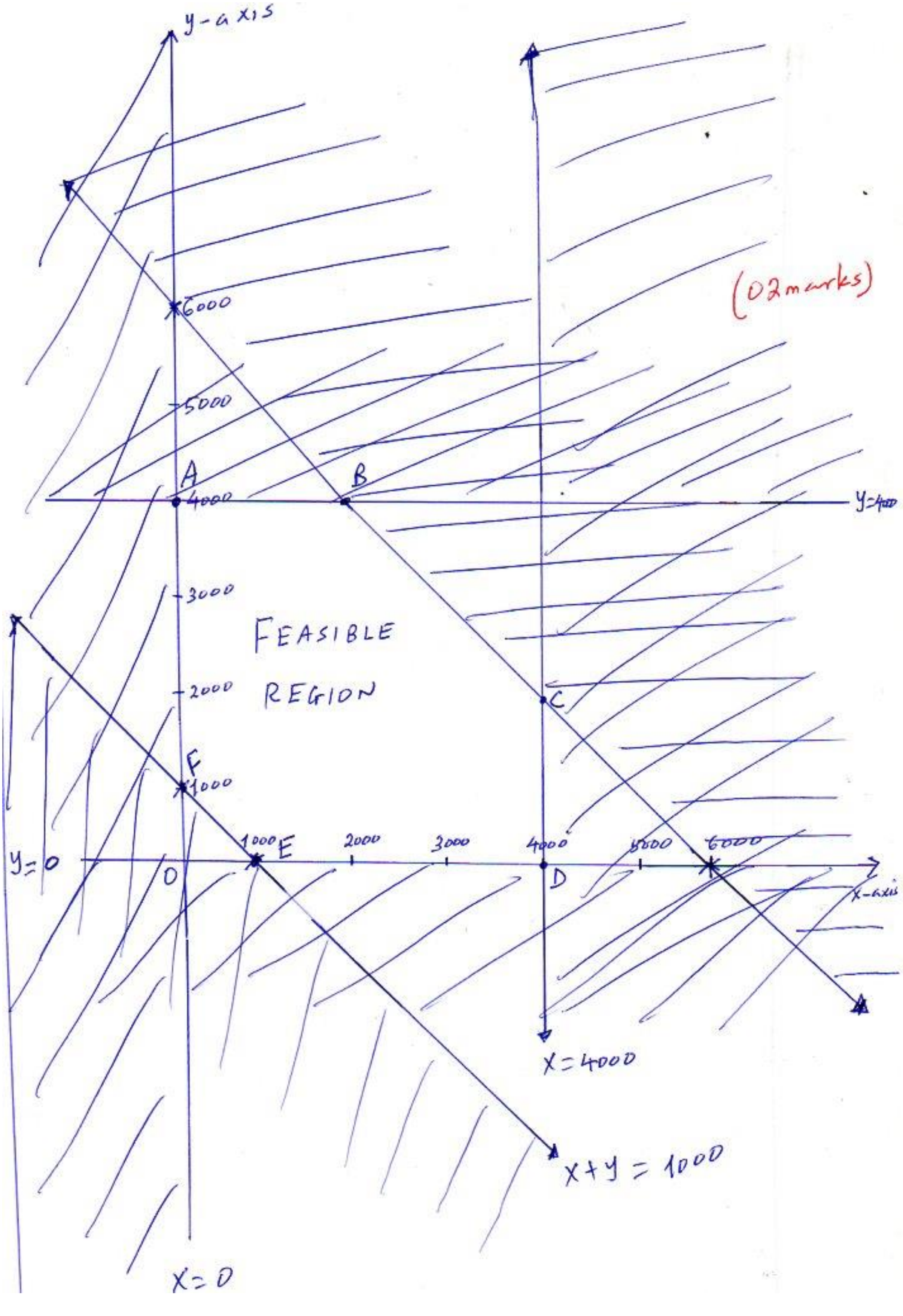
0 packets from A to Q

5,000 packets from A to R

3,000 packets from B to P

4,000 packets from B to Q

0 packets from B to R


y-axis
(02marks)
6000
5000
A
4000
B
y=4000
3000
FEASIBLE
REGION
2000
C
F
1000
y=0
O
1000 E
2000
3000
4000
D
5000
6000
x-axis
x=4000
x+y=1000
x=0

4. a) $Mean = \frac{\sum x}{N}$= 2500

Standard deviation= $\sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$=2000

If all salaries are increased by 2500

New mean =$\frac{\sum X+2500N}{N}$=$\frac{\sum X}{N}$ +2500=25000 +2500=27500

Therefore, new mean is 27500

New S.D= $\sqrt{\sum \frac{(X+2500)^2}{N} - \left(\frac{\sum X}{N} + 2500\right)^2}$

$$New\ \ S.D = \sqrt{\sum \frac{(X^2+5000X+2500^2)}{N} - \left(\left(\frac{\sum X}{N}\right)^2 + 5000\frac{\sum X}{N} + 2500^2\right)}$$

The new standard deviation is 2000

b) Machine A

Mean =$\frac{\sum x}{N} = \frac{2000}{10}$=200

Machine B

Mean=$\frac{\sum x}{N} = \frac{2000}{10}$=200m

Machine A Machine B

X	x-200	$(x-200)^2$	x	x-200	$(x-200)^2$
196	-4	16	192	-8	64
198	-2	4	194	-6	36
198	-2	4	195	-5	25
199	-1	1	198	-2	4
200	0	0	200	0	0
200	0	0	201	1	1
201	1	1	203	3	9
201	1	1	204	4	16
202	2	4	206	6	36
205	5	25	207	7	49
		56			240

Machine A $S^2 = \frac{\sum(X-200)^2}{10} = \frac{56}{10} = 5.6$

S=$\sqrt{5.6} \approx 2.37$

Mchine B $S^2 = \frac{\sum(X-200)^2}{10} = \frac{240}{10} = 24$

S=$\sqrt{24} \approx 4.90$

The S.D for machine Ais 2.37g and the S.D for machine B is 4.90g

Indicating that machine A is more reliable /uniforms

C)

Intervals	*30-39.5*	*40-49.5*	*50-59.5*	*60-69.5*	*70-79.5*	*80-89.5*
Frequencies	*11*	*18*	*24*	*21*	*14*	*12*
Com.frq	*11*	*29*	*53*	*74*	*88*	*100*

Inter-quartile range=$Q_3 - Q_1$

The position of $Q_1 = \left(\frac{N}{4}\right)^{th} = \left(\frac{100}{4}\right)^{th} = 25^{th}$

Q_1 *Class is 40-49.5*

$Q_1 = L_{Q_1} + \left(\frac{\frac{N}{4} - \sum fb}{fw}\right) c$

$$Q_1 = 39.75 + \left(\frac{25 - 11}{18}\right) 10 = 47.53$$

The position of $Q_{3 = \left(\frac{3N}{4}\right)^{th} = \left(\frac{3*100}{4}\right)^{th} = 75^{th}}$ Q_3 $Class\ is\ 70 - 79.5$

$$Q_{3 = L_{Q_3}} + \left(\frac{\frac{3N}{4} - \sum fb}{fw}\right) c$$

$$Q_3 = 69.75 + \left(\frac{75 - 74}{14}\right) 10 = 70.46$$

Therefore, quartile deviation is 70.46-47.53=22.93

5 a) i) Consider L.H.S

$A\nabla(A \cap B)$*……………………………………………………………………given*

$[A - (A \cap B)] \cup [(A \cap B) - A]$*…………..   ……………………by definition of* ∇.

.$[A \cap (A \cap B)'] \cup [(A \cap B) \cap A']$ ……………………………… $by\ definition\ of\ set\ difference.$

$[A \cap (A \cap B)'] \cup [(A \cap B) \cap A']$*………………………………DE Morgan's law.*

$[(A \cap A') \cup (A \cap B')] \cup [A \cap A') \cap (B \cap A')]$ ………..$distributive\ law.$

$[\emptyset \cup (A \cap B')] \cup [\emptyset \cap (B \cap A')]$ ……………..……..$complement\ law.$

(A$\cap$ B'*)* $\cup\ \emptyset$ …………………………………………………..$identity\ law.$

(A-B)………………………………………………………………. by definition of set difference.

Therefore, $A\nabla(A \cap B) = (A - B)$

ii) $[A \cap (A' \cup B)] \cup [B \cap (A' \cup B')]$ ……………………………. *given.*

$[(A \cap A'1) \cup (A \cap B)] \cup [(B \cap A') \cup (B \cap B')]$ …………. *distributive law.*

$[\emptyset \cup (A \cap B)] \cup [(B \cap A') \cup \emptyset]$………………………………..*complement law.*

$(A \cap B) \cup (B \cap A')$ ……………………………………………….*identity law.*

$(A \cap B) \cup (A' \cap B)$ ………………………………………………….. *commutative law.*

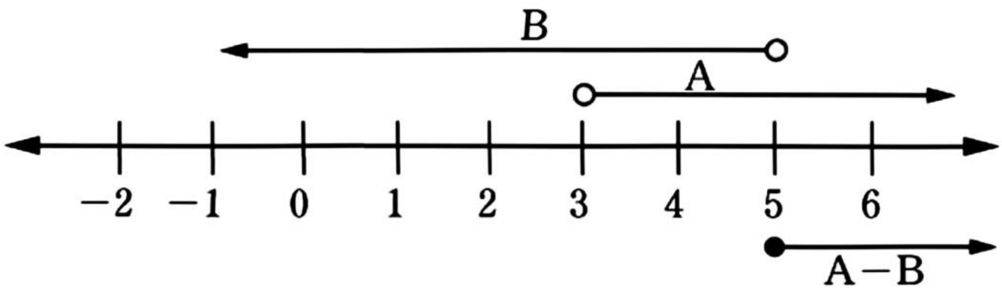
$(A \cup A') \cap B$……………………………………………………………….*Distributive law.*

$\mu \cap B$ ……………………………………………………………… *complement law.*

B……………………………………………………………………………….*Identity law.*

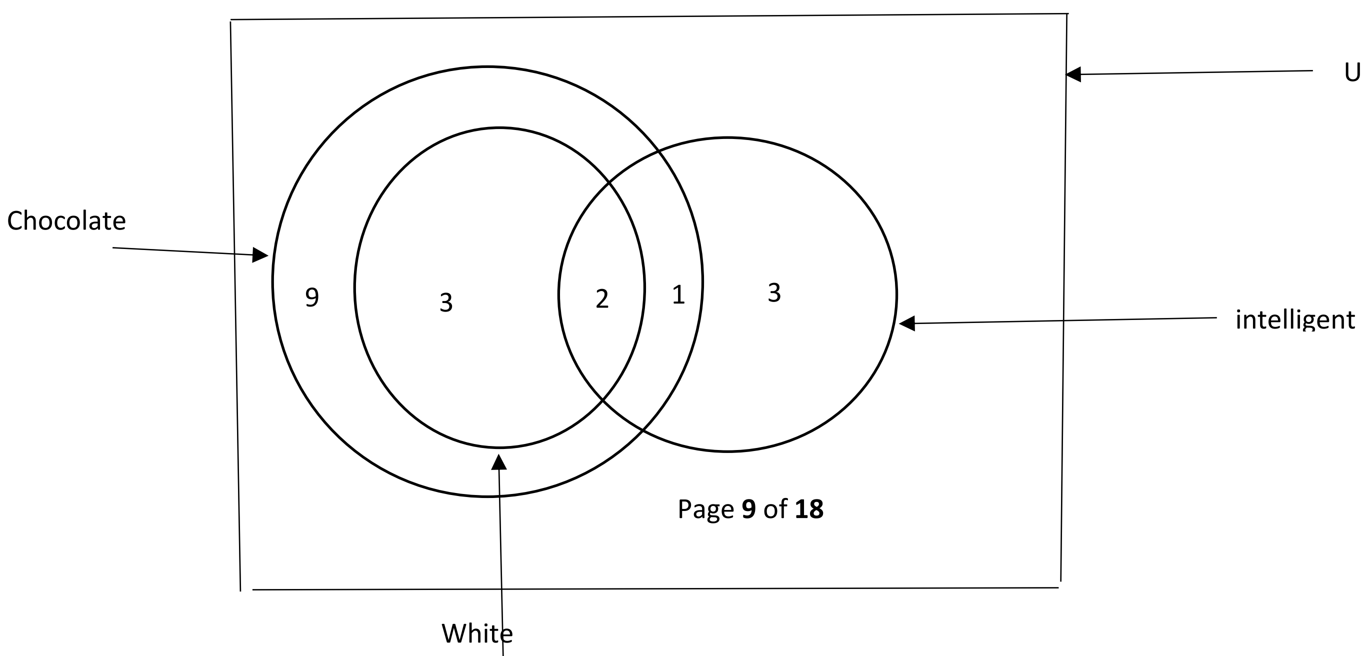
b)

Solution



Therefore, $A - B = \{x : x \geq 5\}$ as represented on the number line.

c)i)The given information can be represented in the Venn diagram as follows;



ii)Number of girls in the class =9+3+2+1+3=18

iii) 3 girls are white but not intelligent.

6. a)i)Condition for f(x) to be an even function

F(-x) = f(x).

ii) Given f(x)=$\frac{px+q}{x+r}$ then f(-x) =$\frac{-px+q}{-x+r}$

For even function f(-x) = f(x)

$\frac{px+q}{x+r} = \frac{-px+q}{-x+r}$.

(px+q)(r-x)=(q-px)(x+r)

Pxr-qx= qx-pxr

2prx=2qx

Pr =q

Then f(x)=$\frac{px+pr}{x+r}$=$\frac{p(x+r)}{x+r}$

F(x)=p hence shown.

b) We have f(x)=$\sqrt[4]{x-1}$ and g(x)=x^4+1

Required to show fog(x)=gof(x)

LHS=$f[g(x)]$=f(x^4+1)

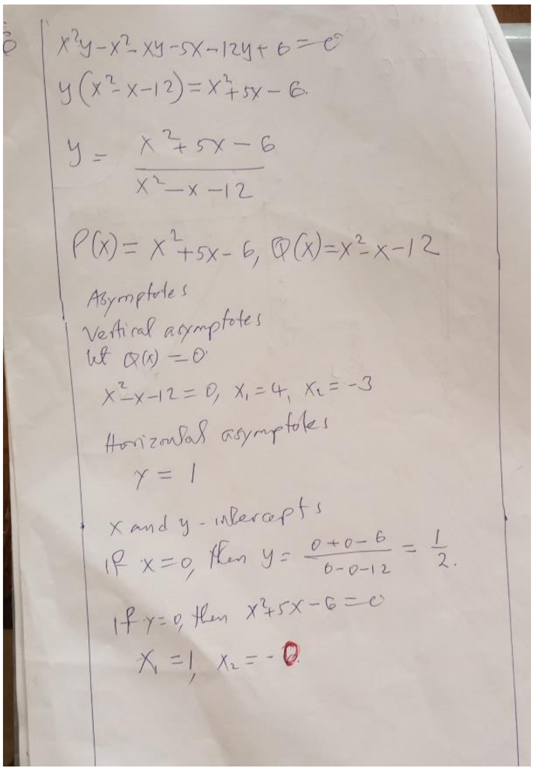
LHS=$\sqrt[4]{X^4+1-1}$=X

Also,

R.H.S=g[$f(x)] = g(\sqrt[4]{x-1}$)

R.H. S=($\left(\sqrt[4]{X-1}\right)^4+1=X$

LHS =RHS. Shown.

c) 

$x^2y - x^2 - xy - 5x - 12y + 6 = 0$

$y(x^2 - x - 12) = x^2 + 5x - 6$

$y = \frac{x^2+5x-6}{x^2-x-12}$

$P(x) = x^2 + 5x - 6$, $Q(x) = x^2 - x - 12$

Asymptotes

Vertical asymptotes

let $Q(x) = 0$

$x^2 - x - 12 = 0$, $x_1 = 4$, $x_2 = -3$

Horizontal asymptotes

$y = 1$

x and y - intercepts

If $x = 0$, then $y = \frac{0+0-6}{0-0-12} = \frac{1}{2}$.

If $y = 0$, then $x^2 + 5x - 6 = 0$

$x_1 = 1$, $x_2 = -6$

$x^2(y-1) - (y+5)x - 6(2y-1) = 0$

$x^2y - x^2 - xy - 5x - 12y + 6 = 0$.

y $(x^2 - x - 12) = x^2$+5x-6

y=$\frac{x^2+5x-6}{x^2-x-12}$,

p(x)=x^2+5x-6, Q(x)=x^2-x-12

Asymptotes

Vertical asymptotes

Let Q(x)=0

$x^2 - x - 12 = 0, x_{1=4}, x_{2=-3}$.

Horizontal asymptotes

Since degree N=degree D

y=1.

X and y-intercepts

If x=0, then y=$\frac{0+0-6}{0-0-12} = \frac{1}{2}$.

If y=0, then x^2+5x-6=0

x_1=1, x_2= -6

Its graph

7. a) Absolute error and Relative errors differ in various aspects namely; definition, determination, size of the quantity and units used to express it.

b) let g=$4\pi^2 \frac{l}{T^2}$

Introducing ln both sides gives

lng=ln4+2lnπ+lnl-2lnT

differentiating each term and maximizing errors gives

$\frac{\Delta g}{g} = 0 + 0 + \frac{\Delta l}{l} + 2\frac{\Delta T}{T}$.

Substituting the values into the equation gives

$$\frac{\Delta g}{g} = 3\% + 2(4\%) = 11\%.$$

Therefore, the percentage error in the determination of g is 11%

c)From v=$\int_a^b y^2\, dx$, but y=$e^{\frac{-1}{2x^2}}$

v =$\pi \int_{-3}^{3} \left(e^{\frac{-1}{2x^2}}\right)^2 dx$

let f(x)=$e^{\frac{-1}{x^2}}$

h=$\frac{b-a}{n} = \frac{3-(-3)}{6}$= 1

x	-3	-2	-1	0	1	2	3
			e^{-1}		e^{1}		

F(x)	$e^{\frac{-1}{9}}$	$e^{\frac{-1}{4}}$		$e^{\frac{1}{0}}$		$e^{\frac{-1}{4}}$	$e^{\frac{-1}{9}}$
Ordinate	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's rule

V=$\frac{h}{3}\pi($

8.a) Given the circle $x^2+y^2-2x+y-5=0$ at $(3,-2)$

slope $=\frac{dy}{dx}$

$\frac{d}{dx}\left(x^2+y^2-2x+y-5=0\right)$

$2x+2y\frac{dy}{dx}-2+\frac{dy}{dx}=0$

$(2y+1)\frac{dy}{dx}=2-2x$

$$\frac{dy}{dx}=\frac{2-2x}{2y+1}$$

at $(3,-2)$

$$M_T=\frac{4}{3}$$

Equation of tangent $\frac{4}{3}=\frac{y+2}{x-3}$

$3y-4x+18=0$

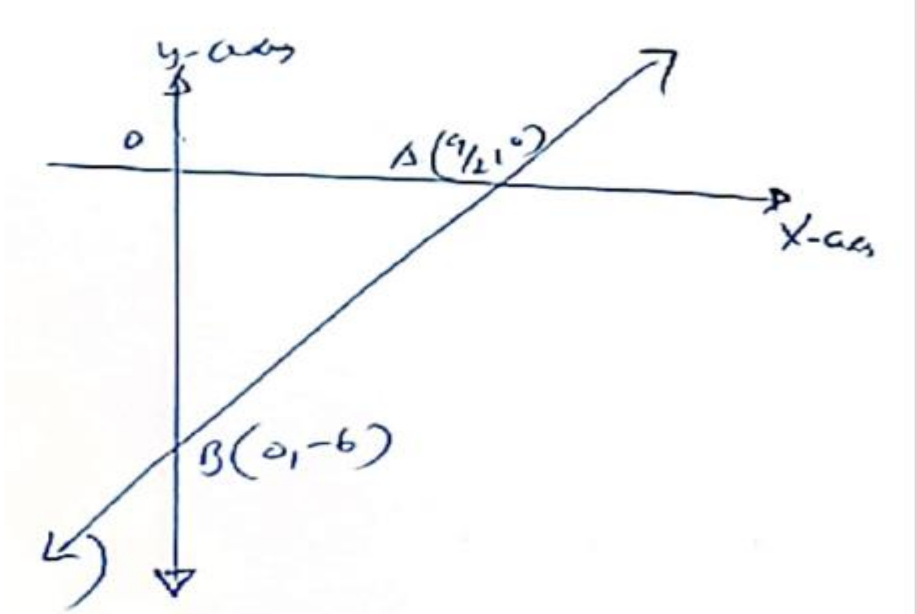
Let A be x-axis, $y=0$

$$(x,y)=\left(\frac{9}{2},0\right)$$

B be at y-axis, x=0

$(x,y)=(0,-6)$

Consider the sketch below



Area of the triangle $=\frac{1}{2}\left|\overline{OA}\right|\left|\overline{OB}\right|$

$\overline{OA}=\frac{9}{2}$ units

$\overline{OB}=6$ units

Area $=\frac{1}{2}\times 6\times\frac{9}{2}$

Area = $(OAB)=\frac{27}{2}$ square units.

b)i) Given a pair of lines $4x^2-24xy+11y^2=0$

by general formula $x=\frac{24y\pm\sqrt{576y^2-176y^2}}{8}$

$$x = \frac{24y \pm 20y}{8}$$

$$8x = 24y \pm 20y$$

For (+ve)

$$2x = 11y$$

$$y = \frac{2}{11}x \text{ …………(i)}$$

Also for (-ve)

$$8x = 4y$$

$$y = 2x \text{ …………..(ii)}$$

From (i) and (ii)

$$M_1 = \frac{2}{11},\ M_2 = 2$$

$$\theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

$$\theta = \tan^{-1}\left(\frac{2 - \frac{2}{11}}{1 + 2 \times \frac{2}{11}}\right)$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\theta = 53^\circ 7'$$

ii) Given the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the circle $x^2 + y^2 = c^2$

$$bx + ay = ab$$

$$y = b - \frac{b}{a}x$$

Substitute to $x^2 + y^2 = c^2$

$$x^2 + \left(b - \frac{b}{a}x\right)^2 = c^2$$

$$x^2 + b^2 - 2\frac{b^2}{a}x + \frac{b^2}{a^2}x^2 = c^2$$

$$\left(1 + \frac{b^2}{a^2}\right)x^2 - 2\frac{b^2}{a}x + \left(b^2 - c^2\right) = 0$$

Since the line $y = b - \frac{b}{a}x$ touches the circle, then $b^2 = 4ac$

$$\frac{4b^4}{a^2} = 4\left(1 + \frac{b^2}{a^2}\right)\left(b^2 - c^2\right)$$

$$\frac{b^4}{a^2} = b^2 - c^2 + \frac{b^4}{a^2} - \frac{b^2c^2}{a^2}$$

$$0 = b^2 - c^2 - \frac{b^2c^2}{a^2}$$

Multiply by a^2 both sides

$$0 = a^2b^2 - a^2c^2 - b^2c^2$$

Divide each term by $a^2b^2c^2$

$$0 = \frac{1}{c^2} - \frac{1}{b^2} - \frac{1}{a^2}$$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$$

c) PQ is divided internally ratio 2:1 by point A (1, 1)

From:

$$A(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

$$(1,1) = \left(\frac{2x_2 + x_1}{3}, \frac{2y_2 + y_1}{3} \right)$$

By comparing

$2x_2 + x_1 = 3 ....(i)$

$2y_2 + y_1 = 3 ....(ii)$

Also

PQ divided externally ratio 5:2 by point B (4, 7) from:

$$B(x, y) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right)$$

$$(4,7) = \left(\frac{5x_2 - 2x_1}{3}, \frac{5y_2 - 2y_1}{3} \right)$$

By comparing

$5x_2 - 2x_1 = 12 ....(iii)$

$5y_2 - 2y_1 = 21 ....(iv)$

Solving (i) and (iii) and (ii) and (iv) simultaneously we get:

$x_1 = -1, y_1 = -3, x_2 = 2, y_2 = 3$

Therefore;

$P(x, y) = (-1, -3)$

$Q(x, y) = (2, 3)$.

9. a) i) let u= $x^5 - 1, du = 5x^4\, dx$,$dx = \frac{du}{5x^4}$

b) Consider

c =$\int e^{ax} \, cosbx \, dx$

by integration by parts

let u=cosbx, du=-bsinbx

dv=$e^{ax}\, dx$,

v= $\frac{1}{a}e^{ax}$,

from $\int u\,dv = uv - \int vdu$

s == $\frac{1}{a}e^{ax}$sinbx -$\frac{b}{a}\int e^{ax}cosbx\,dx$

as= e^{ax}sinbx-b$\int e^{ax}cosbx\,dx$ but c=$\int e^{ax}cosbx\,dx$

as=$e^{ax}sinbx - bc$

Therefore, as + bc=$e^{ax}sinbx$.

c)

10.a) i) $f(x) = lnx$

$f(x+h) = \ln(x+h)$

$From\ first\ principles$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$……

$= \lim_{h\to 0}\frac{ln(x+h)-lnx}{h}$.

$= \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{h}\right)}{h}$

$= \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h}$

From the series expansion of $ln(1+x)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$\ln\left(1+\frac{h}{x}\right) = \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \cdots$

$= \frac{h}{x}(1 - \frac{h}{2x} + \frac{h^2}{3x^2} -$

$f'(x) = \lim_{h\to 0}\frac{\frac{h}{x}(1 - \frac{h}{2x} + \frac{h^2}{3x^2} - \cdots)}{h}$

$=\lim_{h\to 0}\frac{1}{x}(1 - \frac{h}{2x} + \frac{h^2}{3x^2}$As h→0, $1 - \frac{h}{2x} + \frac{h^2}{3x^2} - \cdots) \to 1$

$f'(x) = \frac{1}{x}$.

ii) Given $z = ln\sqrt{x^2+y^2}$

$z = \frac{1}{2}\ln\,(x^2+y^2)$

$\frac{\partial z}{\partial x} = \frac{1}{2}\,(\frac{2x}{x^2+y^2}) = \frac{x}{x^2+y^2}.$

$\frac{\partial z}{\partial y} = \frac{1}{2}\,(\frac{2y}{x^2+y^2}) = \frac{y}{x^2+y^2}.$

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{x}{x^2+y^2}\right) + y\left(\frac{y}{x^2+y^2}\right) = \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}$

$= \frac{x^2+y^2}{x^2+y^2}$

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.$

b)

c) Let r be radius of each tin and h be perpendicular height then the surface area S is given by

S = $2\pi r^2 + 2\pi rh$ but $V = \pi r^2 h$

$128\pi = \pi r^2 h$

$h = \frac{128}{r^2}$

S = $2\pi r^2 + 2\pi r(\frac{128}{r^2})$

S = $2\pi r^2 + \frac{256\pi}{r}$

$\frac{ds}{dr} = 4\pi r - \frac{256\pi}{r^2}.$

For minimum surface area

$\frac{ds}{dr} = 0$

$4\pi r - \frac{256\pi}{r^2} = 0$

$4\pi r = \frac{256\pi}{r^2}$

$4r^3 = 256$

$r^3 = 64$

$r = 4.$

$$h = \frac{128}{r^2}$$

$$h = \frac{128}{4^2}$$

$$h = 8.$$

For minimum surface area each tin should have a radius of 4cm and perpendicular height of 8cm

------------------------------x--------------------------------------x-------------------------