MARKING SCHEME

ADV. MATHS 2

CSSC

1(c) The required number it has four or five digits

$$^5P_4 + {^5P_5} = \frac{5!}{(5-4)!} + \frac{5!}{(5-5)!} = 240$$

Numbers = 240

Alternative

By using table

1^{st}	2^{nd}	3^{rd}	4^{th}
5	4	3	2

or

1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
5	4	3	2	1

Number of numbers $= (5\times4\times3\times2) + (5\times4\times3\times2\times1)$

The number of numbers is 240

1

(b) Let E = student read English Newspaper
S = student read Swahili Newspaper

(i) Probability of neither English nor Swahili
i.e $P(E\cup S)'$

from $P(E\cup S)' = 1 - P(E\cup S)$

$$= 1 - (P(E) + P(S) - P(E\cap S))$$

$$= 1 - (0.7 + 0.3 - 0.2)$$

$$P(E\cup S)' = 0.2$$

(ii) $P(S/E) = \frac{P(E\cap S)}{P(E)} = \frac{0.2}{0.7}$

$$P(S/E) = \frac{2}{7}$$

(iii) $P(E/S) = \frac{P(E\cap S)}{P(S)} = \frac{0.2}{0.3}$

$$P(E/S) = \frac{2}{3}$$

Q1 (c) from $E(x) = \sum x p(x)$

But $p(x) = {}^nC_x p^x q^{n-x}$

$$E(x) = \sum x \left(\frac{n!}{(n-x)!\,x!}\right) p^x q^{n-x}$$

$$E(x) = \sum x \left(\frac{n(n-1)!}{(n-x)!\,x(x-1)!}\right) p p^{x-1} q^{n-x}$$

$$E(x) = np \sum \left(\frac{(n-1)!}{(n-x)!\,(x-1)!} (p^{x-1} q^{n-x})\right)$$

But

$$\sum \left(\frac{(n-1)!}{(n-x)!\,(x-1)!}\right) p^{x-1} q^{n-x} = 1$$

Then

$E(x) = np$

Hence proved.

Mean $-$ Variance $= np - npq$

Mean $-$ Variance $= np(1-q)$

Mean $-$ Variance $= np^2 >$ Variance.

then

Mean $-$ Variance $>$ Variance

2(a) (i) A statement is a sentence that is either true or false. Statements are sentences about facts. Example of statement John's hair is brown, this is a statement because it is either true or not true.

If the sentence expresses an opinion about something instead of verifiable fact about something then it is not statement. etc example; Go to bed

(ii) $2+9=12 \to F$

2 is a factor of 12 $\to T$

$F \leftrightarrow T$

$(F \to T) \land (T \to F)$

$T \land F$

F

The truth value is F

(b) Given $p \to (q \to \sim p)$

Contrapositive $\sim(q \to \sim p) \to \sim p$

Converse of Contrapositive

$\sim p \to \sim(q \to \sim p)$

2b)

p	q	$\sim p$	$\sim q$	$q \to \sim p$	$\sim(q \to \sim p)$	$\sim p \to \sim(q \to \sim p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	T
F	T	T	F	T	F	F
F	F	T	T	T	F	F

The output of the resulting truth table means $p \equiv \sim p \to \sim(q \to \sim p)$

c) Let p – the diagnostic message is stored in the buffer

q – the diagnostic message is transmitted

from the compound statement given:

$$(p \vee q),\ \sim p,\ (p \to q)$$

$$(p \vee q) \wedge \sim p \wedge (p \to q)$$

By using laws of logic

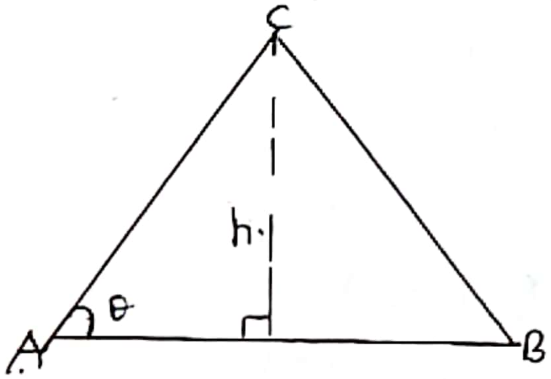
$(p \wedge \sim p) \vee (q \wedge \sim p) \wedge (p \to q)$ --- distributive

$F \vee (q \wedge \sim p)$ --- compliment law.

$(F \vee q) \wedge (F \vee \sim p)$ --- distributive

$q \wedge \sim p$ --- identity.

Q3. (a) i. Consider the diagram above.



$$Area = \frac{1}{2}(\text{base} \times \text{height})$$

$$Area = \frac{1}{2}|\overline{AB}|h.$$

$$\sin\theta = \frac{h}{AC}$$

$$h = |\overline{AC}|\sin\theta$$

$$Area = \frac{1}{2}|\overline{AB}||\overline{AC}|\sin\theta$$

Then;

$$\boxed{Area = \frac{1}{2}\left|\underline{AB} \times \underline{AC}\right|.}$$

03. (ii) Given $A(-2, 3, 1)$ $B(2, 4, -1)$ $C(1, 1, 1)$

$\bar{AB} = \bar{OB} - \bar{OA}$

$\bar{AB} = (2, 4, -1) - (-2, 3, 1)$

$\bar{AB} = (4, 1, -2)$

$\bar{AC} = \bar{OC} - \bar{OA}$

$\bar{AC} = (1, 1, 1) - (-2, 3, 1)$

$\bar{AC} = (3, -2, 0)$

then;

$$\bar{AB} \times \bar{AC} = \begin{vmatrix} i & j & k \\ 4 & 1 & -2 \\ 3 & -2 & 0 \end{vmatrix}$$

$\bar{AB} \times \bar{AC} = i(-4) - j(6) + k(-8-3)$

$\bar{AB} \times \bar{AC} = -4i - 6j - 11k$

$|\bar{AB} \times \bar{AC}| = \sqrt{16 + 36 + 121}.$

$|\bar{AB} \times \bar{AC}| = \sqrt{173}$

But $A = \frac{1}{2} |\bar{AB} \times \bar{AC}|$

$A = \frac{1}{2} (\sqrt{173})$

$A = \frac{\sqrt{173}}{2}$ sqr unit.

03 (c)

$P = 2\underline{a} + \underline{b}$

$Q = \underline{a} - 3\underline{b}$

$m:n = 1:2$

$R = \dfrac{mQ - nP}{m-n}$.

$R = \dfrac{(\underline{a} - 3\underline{b}) - 2(2\underline{a} + \underline{b})}{1-2}$

$R = 3\underline{a} + 5\underline{b}$

Midpoint of RQ

$\dfrac{R+Q}{2} = \dfrac{3\underline{a} + 5\underline{b} + \underline{a} - 3\underline{b}}{2}$

$RQ = \dfrac{4\underline{a} + 2\underline{b}}{2}$

$RQ = 2\underline{a} + \underline{b}$

Midpoint of $RQ = 2\underline{a} + \underline{b} = P$.

Q4 (a) Given; $\arg(w-5+3i) = \arg(5-3i) + \pi/2$.

let $w = x+iy$.

$$\arg(x+iy-5+3i) = \arg(5-3i)+\pi/2$$

$$\arg((x-5)+i(y+3)) = \arg(5-3i)+\pi/2.$$

But $\arg(x+iy) = \tan^{-1}\frac{y}{x}$.

$$\tan^{-1}\left(\frac{y+3}{x-5}\right) = \tan^{-1}(-3/5) + \pi/2$$

$$\tan^{-1}\left(\frac{y+3}{x-5}\right) = -\tan^{-1}(3/5) + \pi/2$$

$$\tan^{-1}\left(\frac{y+3}{x-5}\right) + \tan^{-1}(3/5) = \pi/2.$$

let $A = \tan^{-1}\left(\frac{y+3}{x-5}\right)$

$$\tan A = \frac{y+3}{x-5}$$

let $B = \tan^{-1} 3/5$

$$\tan B = 3/5.$$

Then:

$A+B = \pi/2$, apply tan to both side.

04 (a) $\tan(A+B) = \tan \pi/2$.

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\sin \pi/2}{\cos \pi/2}$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{0}$$

then; $1 - \tan A \tan B = 0$.

$$\tan A \tan B = 1$$

$$\left(\frac{y+3}{x-5}\right)\left(3/5\right) = 1$$

$$\frac{3y+9}{5x-25} = 1$$

$$3y + 9 = 5x - 25$$

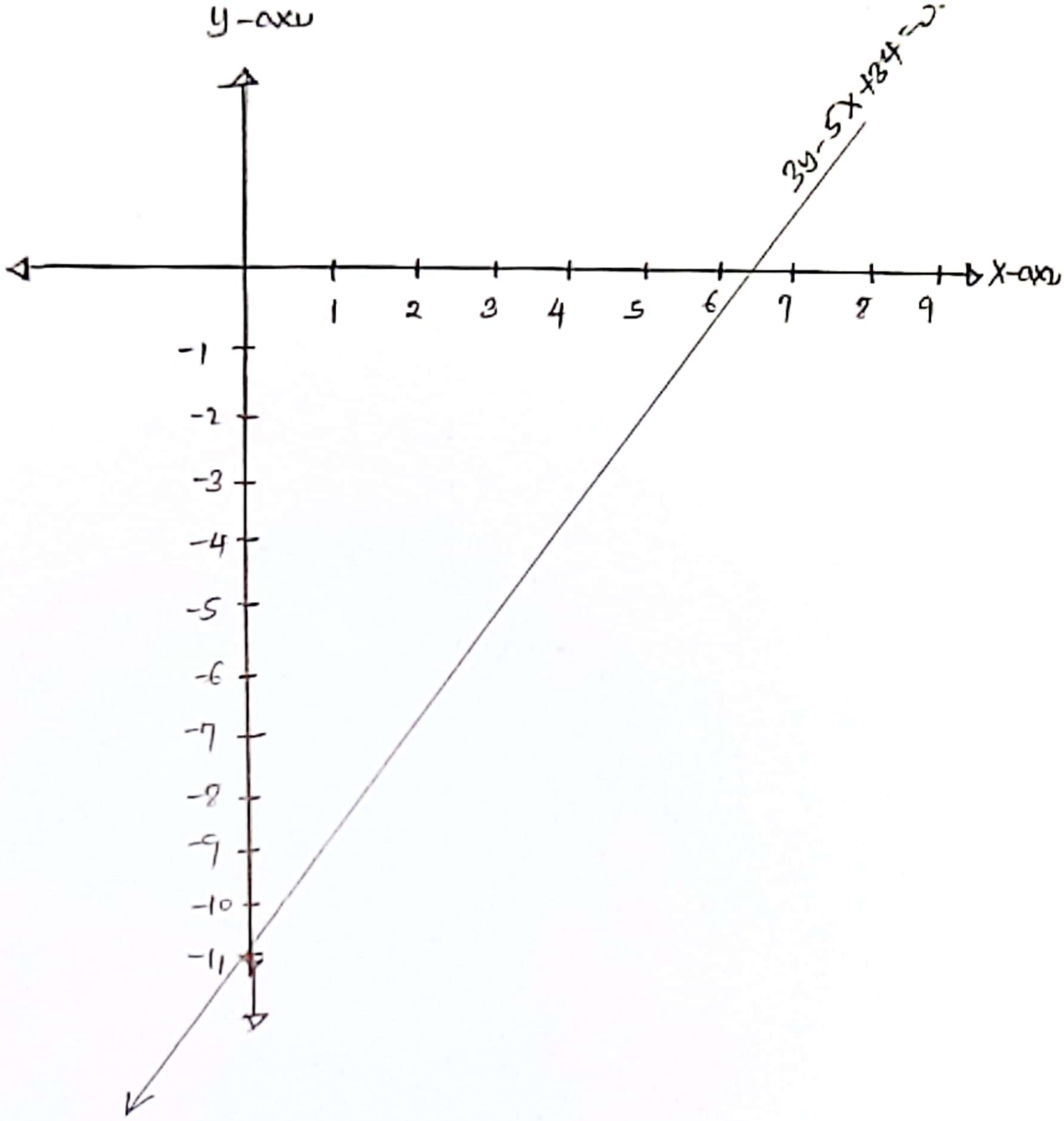
$$3y - 5x + 84 = 0$$

The locus of a complex number is a straight line.

x - intercept $(6{\cdot}8, 0)$

y - intercept $(0 \; -11{\cdot}3)$.

84 (c) To sketch the Graph:



Q4 (b) we have, $\left(z+\frac{1}{z}\right)^5 \left(z-\frac{1}{z}\right)^5$

Recall; $a^n b^n = (ab)^n$

$$\left(z+\frac{1}{z}\right)^n\left(z-\frac{1}{z}\right)^n = \left[\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right]^n$$

$$= (z^2 - z^{-2})^5$$

But;

$$(a+b)^5 = a^5 + {}^5C_1 a^4 b + {}^5C_2 a^3 b^2 + {}^5C_3 a^2 b^3 + {}^5C_4 ab^4 + b^5$$

where $a = z^2$ and $b = -z^{-2}$

Then:

$$(z^2 - z^{-2})^5 = (z^2)^5 + 5(z^2)^4(-z^{-2}) + 10(z^2)^3(-z^{-2})^2 + 10(z^2)^2(-z^{-2})^3$$

$$+ 5(z^2)(-z^{-2})^4 + (-z^{-2})^5$$

then simplify.

$$(z^2 - z^{-2})^5 = z^{10} - 5z^6 + 10z^2 + 10z^{-2} + 5z^{-6} - z^{-10}$$

$$(z^2 - z^{-2})^5 = z^{10} - z^{-10} - 5(z^6 - z^{-6}) + 10(z^2 - z^{-2})$$

$$(2i\sin 2\theta)^5 = 2i\sin 10\theta - 5(2i\sin 6\theta) + 10(2i\sin 2\theta)$$

$$(2i)^5(2\sin\theta\cos\theta)^5 = 2i(\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta)$$

$$2^{10}(\sin^5\theta\cos^5\theta) = 2(\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta)$$

$$\sin^5\theta\cos^5\theta = \frac{1}{2^9}(\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta)$$

Hence shown.

04 (b) Now, $\int_0^{\pi/2} \sin^5\theta \cos^5\theta = \frac{1}{2^9}\int_0^{\pi/2} (\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta)\, d\theta$

$$= \frac{1}{2^9}\left[-\frac{1}{10}\cos 10\theta + \frac{5}{6}\cos 6\theta - 5\cos 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2^9}\left[0 - \left(-\frac{1}{10} + \frac{5}{6} - 5\right)\right].$$

$$= \frac{1}{120}.$$

then; $\int_0^{\pi/2} \sin^5\theta \cos^5\theta \, d\theta = \frac{1}{2^9}\int_0^{\pi/2} (\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta)\, d\theta = \frac{1}{120}.$

04 (c) Required to show that

$$\tan 4\theta = \frac{4t-4t^3}{t^4-6t^2+1}, \text{ when } t=\tan\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

Using de' moivers theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

then;

$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$$

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

Compare real and imaginary part.

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad \text{---(I)}$$

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \quad \text{---(II)}$$

divide eqn (II) to eqn (I)

$$\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\sin^4\theta - 6\cos^2\theta\sin^2\theta + \cos^4\theta}$$

divide by $\cos^4\theta$ at both numerator and denominator

Q4. (c) $\tan 4\theta = \dfrac{\dfrac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\sin^4\theta}{\cos^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \dfrac{\cos^4\theta}{\cos^4\theta}}$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$$

But $t = \tan\theta$

$$\tan 4\theta = \frac{4t - 4t^3}{t^4 - 6t^2 + 1}$$

Hence shown.

To find roots of $t^4 + 2t^3 - 6t^2 - 2t + 1 = 0$.

Re-arrange.

$$t^4 - 6t^2 + 1 = 2t - 2t^3.$$

$$t^4 - 6t^2 + 1 = 2(t - t^3)$$

$$\frac{t^4 - 6t^2 + 1}{t - t^3} = 2.$$

But; $\dfrac{t^4 - 6t^2 + 1}{t - t^3} = \tan 4\theta$

then; $\tan 4\theta = 2.$

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04 (c) $4\theta = \tan^{-1}(2)$.

from general formular of tan

$$\theta = \pi n + \alpha.$$

$$4\theta = \pi n + 63.4, \quad n = 0, 1, 2, \dots n.$$

$$\theta = \frac{\pi}{4} n + \frac{63.4}{4}$$

when $n=0$ $\quad \theta = 15.85$ — — — 1st root

$n = 1$ $\quad \theta = 60.85$ — — 2nd root

$n = 2$ $\quad \theta = 105.85$ — — 3rd root

$n = 3$ $\quad \theta = 150.85$ — — 4th roots

But $\tan\theta = t$.

at $\theta = 15.85$ $\quad t = \tan(15.85) = 0.28$

$\theta = 60.85, \; t = 1.79$

$\theta = 105.85 \quad t = -3.52$

$\theta = 150.85, \; t = -0.56$.

Hence the roots are. $t = 0.28, \; t = 1.79, \; t = -3.52, \; t = -0.56$

Q5. (a) $\sin 3x \cos 3x - \cos^2 2x + \frac{1}{2} = 0$ ---(1) eqn

soln.

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x)$$

substitute at --(1) eqn

$$\sin 3x \cos 3x - \frac{1}{2} - \frac{1}{2}\cos 4x + \frac{1}{2} = 0$$

$$\sin 3x \cos 3x - \frac{1}{2}\cos 4x = 0$$

$$\sin 3x \cos 3x = \frac{1}{2}\cos 4x$$

$$2\sin 3x \cos 3x = \cos 4x$$

$$\sin 6x = \cos 4x$$

$$\cos 4x = \sin\left(\pi/2 - 4x\right)$$

then

$$\sin 6x = \sin\left(\pi/2 - 4x\right)$$

$$\sin 6x = \sin\left(\pi/2 - 4x\right)$$

$$6x = \sin^{-1}\left(\sin\left(\pi/2 - 4x\right)\right)$$

$$6x = \pi/2 - 4x$$

from general formular for sine $\theta = \pi n + (-1)^n \alpha$.

$$6x = \pi n + (-1)^n\left(\pi/2 - 4x\right)$$

$$x = \pi/6\, n + (-1)^n\left(\pi/2 - 4x\right)1/6$$

$$x = \pi/6\, n + (-1)^n\left(\pi/2 - 4x\right)1/6.$$

05. (b) To express $\cos 5\theta$ in term of $\cos\theta$

$$\cos(5\theta) = \cos(2\theta + 3\theta).$$

$$\cos(2\theta+3\theta) = \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta$$

$$\cos(2\theta+3\theta) = (2\cos^2\theta - 1)(4\cos^3\theta - 3\cos\theta) - 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta)$$

$$\cos(2\theta+3\theta) = 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 6\sin^2\theta\cos\theta + 8\sin^4\theta\cos\theta$$

$$\cos(2\theta+3\theta) : 8\cos^5\theta - 10\cos^3\theta + 3\cos\theta - 6\cos\theta + 6\cos^3\theta + 8(1 - 2\cos^2\theta + \cos^4\theta)\cos\theta$$

$$\cos(2\theta+3\theta) = 8\cos^5\theta - 4\cos^3\theta - 3\cos\theta + 8\cos\theta - 16\cos^3\theta + 8\cos^5\theta$$

$$\cos(2\theta+3\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

for $\theta = 18$

$$\cos 5(18) = 16\cos^5 18 - 20\cos^3 18 + 5\cos 18$$

let $16\cos^5 18 - 20\cos^3 18 + 5\cos 18 = 0$

$$\cos 18\left[(16\cos^4 18 - 20\cos^2 18 + 5)\right] = 0$$

But $\cos 18 \neq 0$

$$16\cos^4 18 - 20\cos^2 18 + 5 = 0$$

then;

$$\cos^2 18 = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32} = \frac{20 \pm \sqrt{80}}{32}$$

$$\cos^2 18 = \frac{20 \pm 4\sqrt{5}}{32} = 2\sqrt{(10 \pm 2\sqrt{5})}$$

05 (c) $3\sin^2\theta + \sqrt{3}\sin\theta\cos\theta + 4\cos^2\theta$

to express in form of $a + b\cos(2\theta - \alpha)$

$$\frac{3}{2}(1-\cos 2\theta) + \frac{\sqrt{3}}{2}\sin 2\theta + \frac{4}{2}(1+\cos 2\theta)$$

$$\frac{3}{2} - \frac{3}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta + 2 + 2\cos 2\theta$$

$$\frac{7}{2} + \frac{1}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta$$

from; $b\cos(2\theta - \alpha) = (\cos 2\theta\cos\alpha + \sin 2\theta\sin\alpha)b$

then;

compare with $\frac{1}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta$

$$\cos 2\theta\cos\alpha = \frac{1}{2}\cos 2\theta$$

$$b\cos\alpha = \frac{1}{2} \quad ----(I).$$

$$\sin 2\theta\sin\alpha = \frac{\sqrt{3}}{2}\sin 2\theta$$

$$b\sin\alpha = \frac{\sqrt{3}}{2} \quad ---(II).$$

Square eqn (I) and (II) then add.

$$b^2 \cos^2\alpha + \sin^2\alpha = \frac{1}{4} + \frac{3}{4} = 1$$

$b^2 = 1$, $b = \pm 1$, take magnitude of b

$b = 1$

05 (c).

divide eqn (ii) to eqn (i).

$$\frac{b\sin\alpha}{b\cos\alpha} = \sqrt{3}/2 \cdot 2$$

$$\tan\alpha = \sqrt{3}.$$

$$\alpha = \tan^{-1}(\sqrt{3})$$

$$\alpha = \frac{\pi}{3}.$$

then;

$$\frac{1}{2}\cos 2\theta + \sqrt{3}/2 \sin 2\theta = \cos(2\theta - \pi/3)$$

Add $7/2$ at both side.

$$7/2 + 1/2 \cos 2\theta + \sqrt{3}/2 \sin 2\theta = 7/2 + \cos(2\theta - \pi/3).$$

hence

$$3\sin^2\theta + \sqrt{3}\sin\theta\cos\theta + 4\cos^2\theta = \frac{7}{2} + \cos(2\theta - \pi/3)$$

Q5 (d)

$\tan(3x - \pi/4) = \tan x$, $(3x - \frac{\pi}{4}) = \tan^{-1}(\tan x)$

$$x = \frac{\pi(4n+1)}{8}$$

$$3x - \pi/4 = \tan^{-1}(\tan x)$$

$$3x - \pi/4 = x.$$

from, General equation of tan

$$x = \pi n + \alpha.$$

$$3x - \pi/4 = \pi n + x.$$

$$3x = \pi n + x + \pi/4$$

$$2x = \pi n + \pi/4$$

$$x = \pi/2\, n + \pi/8$$

$$x = \frac{\pi(4n+1)}{8}$$

Hence proved.

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Q5 (e) $\tan A + \tan B = P$ and $\tan A \tan B = q$.

Required Value of $\cos(2A+2B)$

But

$\cos(2A+2B) = \cos 2A \cos 2B - \sin 2A \sin 2B$

$$\cos(2A+2B) = \left(\frac{1-\tan^2 A}{1+\tan^2 A}\right)\left(\frac{1-\tan^2 B}{1+\tan^2 B}\right) - \left(\frac{2\tan A}{1+\tan^2 A}\right)\left(\frac{2\tan B}{1+\tan^2 B}\right)$$

$$\cos(2A+2B) = \frac{1-\tan^2 B-\tan^2 A+\tan^2 A\tan^2 B}{1+\tan^2 B+\tan^2 A+\tan^2 A\tan^2 B} - \frac{4\tan A\tan B}{(1+\tan^2 A)(1+\tan^2 B)}$$

$$\cos(2A+2B) = \frac{1-\tan^2 B-\tan^2 A+\tan^2 A\tan^2 B - 4\tan A\tan B}{1+\tan^2 B+\tan^2 A+\tan^2 A\tan^2 B}.$$

But $a^2+b^2 = (a+b)^2 - 2ab$

then $\tan^2 B + \tan^2 A = (\tan B + \tan A)^2 - 2\tan A\tan B$.

$$\cos(2A+2B) = \frac{1-\left((\tan B+\tan A)^2 - 2\tan A\tan B\right) + \tan^2 A\tan^2 B - 4\tan A\tan B}{1+(\tan B+\tan A)^2 - 2\tan A\tan B + \tan^2 A\tan^2 B}$$

then;

$$\cos(2A+2B) = \frac{1-((P^2-2q)) + q^2 - 4q}{1+P^2-2q+q^2}$$

6 (a) Let the roots be α, β and γ

we have $ax^3 + bx^2 + cx + d = 0$

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

But given that

$$\alpha = \beta + \gamma \quad \text{----(i)}$$

Sum of roots

$$\alpha + \beta + \gamma = -b/a$$

but $\alpha = \beta + \gamma$

$$2(\beta + \gamma) = -b/a$$

$$\beta + \gamma = \frac{-b}{2a} \quad \text{----(ii)}$$

Sum of product of roots

$$\alpha\beta + \alpha\gamma + \beta\gamma = c/a \quad \text{---(iii)}$$

product of roots

$$\alpha\beta\gamma = -d/a,$$

from equation (i) $\alpha = \beta + \gamma$

$$(\beta + \gamma)(\beta\gamma) = -d/a$$

$$\left(\frac{-b}{2a}\right)\beta\gamma = -d/a$$

$$\beta\gamma = \frac{2d}{b}$$

6(a) Also from equ (iii)

$$\alpha(\beta+\gamma)+\beta\gamma = c/a$$

But $\alpha = (\beta+\gamma) = \dfrac{-b}{2a}$

$$\left(\frac{-b}{2a}\right)\left(\frac{-b}{2a}\right) + \frac{2d}{b} = c/a$$

$$\frac{b^2}{4a^2} + \frac{2d}{b} = c/a$$

$$b^3 + 8a^2d = 4abc$$

$\therefore$ The relation between a, b, c and d

is $b^3 + 8a^2d = 4abc$

(b) We have

$T_2 = 240,\ T_3 = 720,\ T_4 = 1080$

From $T_{r+1} = {}^nC_r\, a^{n-r} b^r$

$$T_2 = 240 \rightarrow {}^nC_1 X^{n-1} a = 240$$

$$nX^{n-1}a = 240 \quad ----(i)$$

$$T_3 = 720,\ {}^nC_2 X^{n-2} a^2 = 720$$

6b

$$\frac{n(n-1)}{2!} x^{n-2} a^2 = 720 \quad \text{------} \text{(ii)}$$

$$T_4 = 1080 \rightarrow {}^nC_3 x^{n-3} a^3 = 1080$$

$$\frac{n(n-1)(n-2)}{6} x^{n-3} a^3 = 1080 \quad \text{---} \text{(iii)}$$

From eqn (ii)

$$\frac{a(n-1)}{2x}\left(n x^{n-1} a\right) = 720$$

$$\frac{a(n-1)}{2x}(240) = 720 \rightarrow \frac{a(n-1)}{2x} = 3$$

$$\frac{a(n-1)}{2x} = 3 \rightarrow \frac{a(n-1)}{6} = x \quad \text{---} \text{(iv)}$$

from equ (iii)

$$\frac{a(n-2)}{3x}\left(\frac{n(n-1)}{2} x^{n-2} a^2\right) = 1080$$

$$\frac{a(n-2)}{3x}(720) = 1080, \quad \frac{a(n-2)}{3x} = \frac{3}{2}$$

$$\frac{2a(n-2)}{9} = x \quad \text{-----} \text{(v)}$$

Since eqn (iv) and (v) are equal

Then

$$\frac{a(n-1)}{6} = \frac{2a(n-2)}{9}$$

$$9(n-1) = 12(n-2)$$

solve for n

$$\underline{\underline{n = 5}}$$

from eqn (iv) into (i)

$$\frac{a(n-1)}{6} = x, \quad x = \frac{3}{2}a$$

$$n\left(\frac{2}{3}a\right)^{5-1} a = 240$$

$$a^5 = 48\left(\frac{3}{2}\right)^4$$

$$\underline{\underline{a = 3}}$$

$$X = \frac{2}{3} \times 3, \quad X = 2$$

Therefore.

$$\underline{\underline{n = 5, \; X = 2 \text{ and } a = 3}}$$

6(c) There is typing error, systems of equations should be:

$x+4y+3=3z$, $-10y+x+7z=13$

and $x-2y+z=3$

Marks can be shifted or if the student did it you can give him/her the marks

(e) Consider

$$\sqrt[4]{1+x}+\sqrt[4]{1-x}=(1+x)^{1/4}+(1-x)^{1/4} \quad ---(i)$$

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots$$

$$(1+x)^{1/4}=1+\frac{1}{4}x-\frac{3}{32}x^2+\frac{7}{128}x^3 \quad ---(ii)$$

$$(1-x)^{\frac{1}{4}}=1-\frac{1}{4}x-\frac{3}{32}x^2-\frac{7}{128}x^3---- \quad -(iii)$$

Substitute eqn (ii) and (iii) to eqn (i)

$$\sqrt[4]{1+x}+\sqrt[4]{1-x}=\left(1+\frac{1}{4}x-\frac{3}{32}x^2+\frac{7}{128}x^3+\cdots\right)+\left(1-\frac{1}{4}x-\frac{3}{32}x^2-\frac{7}{128}x^3\right)$$

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6(c)

$$\sqrt[4]{1+x} + \sqrt[4]{1-x} = 2 - \frac{3}{16}x^2 \quad ----(iv)$$

Equate eqn (iv) with

$$\sqrt[4]{1+x} + \sqrt[4]{1-x} = m - px^2$$

The value of $m = 2$ and $p = \frac{3}{16}$

Hence put $x = \frac{1}{16}$ into $\sqrt[4]{1+x} + \sqrt[4]{1-x} = 2 - \frac{3}{16}x^2$

$$\sqrt[4]{1+\frac{1}{16}} + \sqrt[4]{1-\frac{1}{16}} = 2 - \frac{3}{16}\left(\frac{1}{16}\right)^2$$

$$\sqrt[4]{\frac{17}{16}} + \sqrt[4]{\frac{15}{16}} = 2 - \frac{3}{16^3}$$

$$\frac{1}{2}\left(\sqrt[4]{17} + \sqrt[4]{15}\right) = 2 - \frac{3}{16^3}$$

$$\sqrt[4]{17} + \sqrt[4]{15} = 2\left(2 - \frac{3}{16^3}\right) \simeq 3.9985$$

Hence $\sqrt[4]{17} + \sqrt[4]{15} \simeq 3.9985$ shown

6(d) Consider the expression

$$x^3 + cx^2 - 2cx + 4$$

Divisor $x-1=0 \rightarrow x=1$

1	1	c	-2c	4
		1	c+1	-c+1
	1	c+1	-c+1	-c+5

If $x-1$ is a factor of $P(x)$, the remainder should be equal to zero

$-c+5=0$, $c=5$

Therefore the value of $c=5$

7

@ Given $y = A \sin^{-1} x$ where A is an arbitrary constant.

$$y = \frac{A}{\sin x}$$

$$\frac{dy}{dx} = \frac{A \cos x}{\sin^2 x}$$

but $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$\frac{dy}{dx} = \frac{-2A\cos x}{1 - \cos 2x}$$

$$\frac{d^2y}{dx^2} = ?$$

let $u = -2A\cos x$

$$\frac{du}{dx} = 2A\sin x$$

$$v = 1 - \cos 2x$$

$$\frac{dv}{dx} = 2\sin 2x$$

from $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$= \frac{(1-\cos 2x)(2A\sin x) - (-2A\cos x)(2\sin 2x)}{(1-\cos 2x)^2}$$

But $(1-\cos 2x)^2 = 4\sin^4 x$

$$\frac{d^2y}{dx^2} = \frac{2A\sin x - 2A\sin x\cos 2x + 4A\cos x\sin 2x}{4\sin^4 x}$$

$$= \frac{2A\sin x - 2A\sin x\cos 2x + 4A\cos x\sin x\cos x}{4\sin^4 x}$$

$$= \frac{2A - 2A\cos 2x + 4A\cos x\, 2\cos x}{4\sin^3 x}$$

$$= \frac{2A - 2A\cos 2x + 8\cos^2 x}{4\sin^3 x}$$

$$\frac{d^2y}{dx^2} = \frac{A - 2\cos 2x + 4\cos^2 x}{4\sin^3 x}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

7 (b) (i) $x^2\frac{dy}{dx} + 2xy = \cos x$

$$\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{\cos x}{x^2}$$

$$IF = e^{\int P dx}$$

$$IF = e^{\int \frac{2}{x} dx} = e^{2\int \frac{dx}{x}} = e^{2\ln x}$$

$$IF = e^{\ln x^2} = x^2$$

$$x^2\left(\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2}\right)$$

$$x^2\frac{dy}{dx} + 2xy = \cos x$$ this is exact differential equation

$$\int d(x^2y) = \int \cos x \, dx$$

$$x^2y = \sin x + c$$

$$y = \frac{\sin x + c}{x^2}$$

7 (b) (ii) $\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$

let $P = \frac{dy}{dx}$

$P^2 - 5P + 6 = 0$

factorize, Then $P = 2$, $P = 3$

but $P = \frac{dy}{dx}$

$\frac{dy}{dx} = 2$, $\quad dy = 2dx$

$\int dy = \int 2dx$

$y = 2x + c_1$

Also $\frac{dy}{dx} = 3$, $\quad dy = 3dx$

$\int dy = \int 3dx$

$y = 3x + c_2$

Therefore, $y = 2x + c_1$, $y = 3x + c_2$

where c_1 and c_2 are constants

7 (c) Required to show that the solution to the differential equation $\frac{d^2x}{dt^2} = -4x$

$$\frac{d^2x}{dt^2} + 4x = 0$$

from

$$\frac{d^2x}{dt^2} + w^2x = 0$$

$$w^2 = 4, \quad w = \pm 2$$

Therefore

$$x = A\sin(2t + \theta)$$

$$x = A\cos(2t + \alpha)$$

(d) Given $2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 6y = 3e^{2x}$

Consider Complimentary part

$$2m^2 - 4m - 6 = 0$$

$$m = 3, \quad m = -1$$

from $y = Ae^{\alpha x} + Be^{\beta x}$.

$$y_{c.f} = Ae^{-x} + Be^{3x}$$

Also consider the particular integral part

$$y = ke^{2x}$$

$$\frac{dy}{dx} = 2ke^{2x}$$

$$\frac{d^2y}{dx^2} = 4ke^{2x}$$

from

$$2\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 6y = 3e^{2x}$$

$$2(4ke^{2x}) - 4(2ke^{2x}) - 6(ke^{2x}) = 3e^{2x}$$

$$8ke^{2x} - 8ke^{2x} - 6ke^{2x} = 3e^{2x}.$$

$$-6k = 3$$

$$k = -\tfrac{1}{2}$$

$$\therefore y_{P.I} = -\tfrac{1}{2}e^{2x}$$

General solution $y = y_{c.f} + y_{P.I}$

$$y = Ae^{-x} + Be^{3x} - \tfrac{1}{2}e^{2x}$$

7(e) (i) from $\frac{d\theta}{dt} \propto (\theta - \theta_R)$

$$\frac{d\theta}{dt} = k(\theta - \theta_R)$$

$$\frac{d\theta}{\theta - \theta_R} = k\,dt$$

$$\int_{T_0}^{T} \frac{d\theta}{\theta - \theta_R} = \int k\,dt$$

$$\ln(\theta - \theta_R)\Big|_{T_0}^{T} = kt$$

$$\ln(T - \theta_R) - \ln(T_0 - \theta_R) = kt$$

$$\ln\left(\frac{T - \theta_R}{T_0 - \theta_R}\right) = kt$$

$$\frac{T - \theta_R}{T_0 - \theta_R} = e^{kt}$$

$$e^{kt} = \frac{T - \theta_R}{T_0 - \theta_R}$$

$$e^{kt} = \frac{40 - 5}{60 - 5}$$

$$e^{kt} = \frac{35}{55}$$

$$e^{4t} = 0.636$$

$K = -0.09039$

$$\ln\left(\frac{T-5}{55}\right) = -0.0939 \times 15$$

$$\ln\left(\frac{T-5}{55}\right) = 1.3559$$

$$\frac{T-5}{55} = e^{1.3559}$$

$$= 0.2577$$

$$T-5 = 14.1735$$

$$\underline{\underline{T = 19.1735\,°C}}$$

7(e) (ii) when $T = 10°C$, $t = ?$

$$\ln\left(\frac{T-5}{T_0-5}\right) = kt$$

$$\ln\left(\frac{10-5}{60-5}\right) = kt$$

solve for t

$\underline{\underline{t = 26.528 \text{ minutes}}}$

8. (a) Given $r = \dfrac{6}{1+2\sin\theta}$

$r = \sqrt{x^2+y^2}$, $\sin\theta = \dfrac{y}{r}$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{6}{1+\frac{2y}{\sqrt{x^2+y^2}}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{6\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}+2y}$$

$$\sqrt{x^2+y^2} + 2y = 6$$

$$\sqrt{x^2+y^2} = 6-2y$$

$$x^2+y^2 = (6-2y)^2$$

$$x^2+y^2 = 36-24y+4y^2$$

$$x^2-3y^2+24y = 36$$

$$x^2-3((y-4)^2-16) = 36$$

$$x^2-3(y-4)^2+48 = 36$$

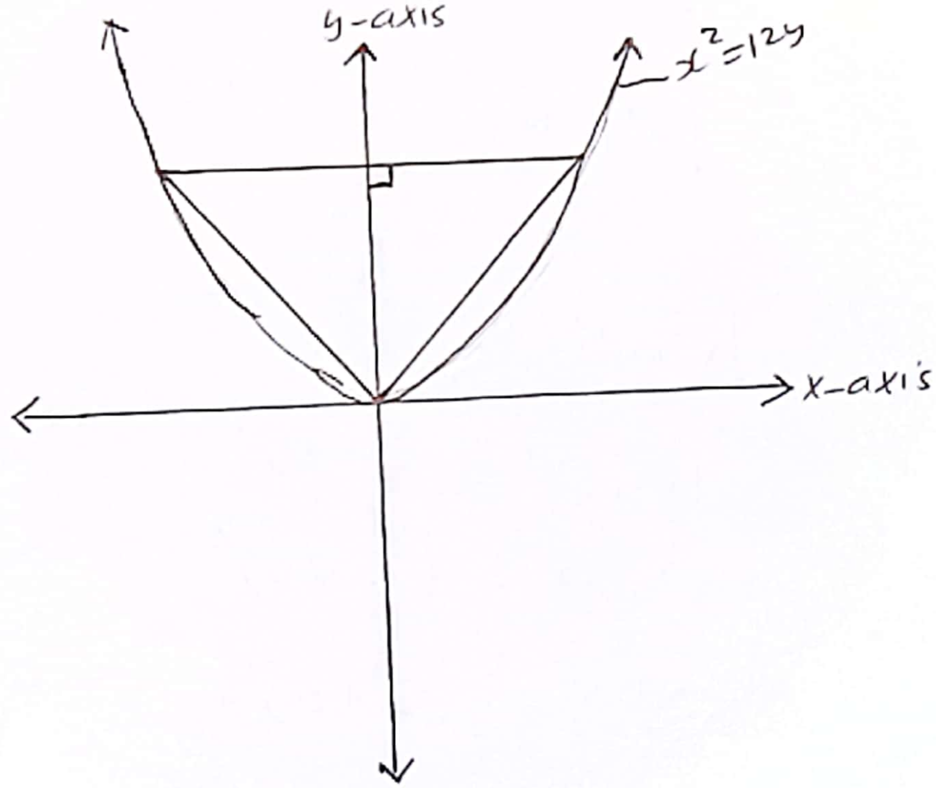
$$\frac{(y-4)^2}{4} - \frac{x^2}{12} = 1$$

This is the equation of the translated hyperbola

8 (5) Required to find area of the triangle formed by lines joining the vertex of the hyperbola $x^2 = 12y$ to the ends of its latus rectum

Solution.

Consider the sketch below



$x^2 = 12y$, compare with $x^2 = 4by$

$b = 3$

Length of latus rectum $L = 4b = 12$

$\text{Area} = \frac{1}{2} L b$

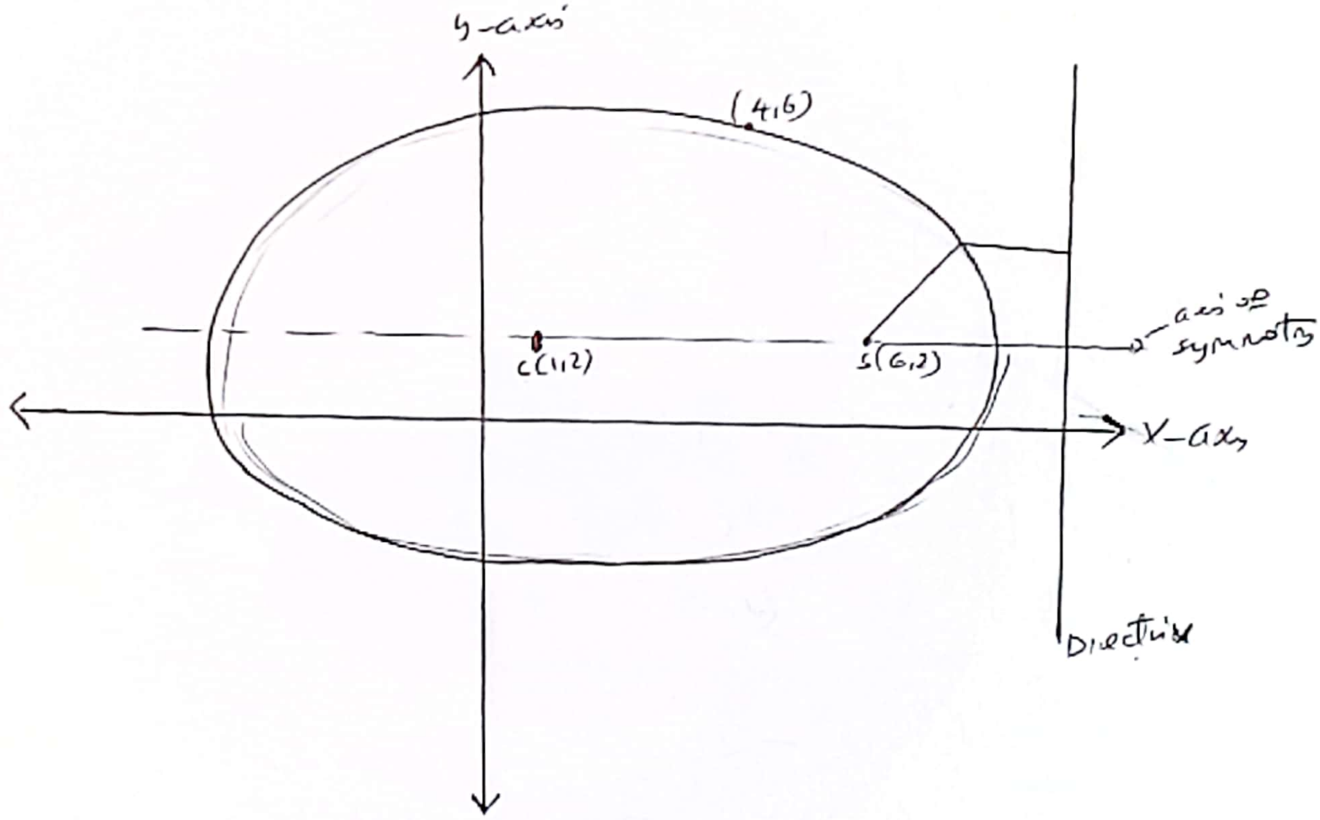
$\text{Area} = \frac{1}{2} \times 12 \times 3 = 18$ square units

Therefore area of triangle is 18 sq. units

8 (c) Required to find equation of an ellipse with centre (1,2), focus (6,2) and passes through (4,6)

Solution

Consider the sketch below



centre $(h,k) = (1,2)$

$h=1$, $k=2$.

focus $(6,2) = (ae+h, k)$

$ae + h = 6$

$ae + 1 = 6$, $ae = 5$

from

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$h = 1$, $k = 2$

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

Passes through $(4,6)$

$$\frac{9}{a^2} + \frac{16}{b^2} = 1$$

$$16a^2 + 9b^2 = a^2b^2 \quad --- (i)$$

Also

$$b^2 = a^2(1-e^2)$$

$$b^2 = a^2 - a^2e^2$$

But $ae = 5$

$$b^2 = a^2 - 25$$

$$a^2 = b^2 + 25 \quad --- (ii)$$

Substitute eqn (ii) to eqn (i)

$$16(b^2+25) + 9b^2 = (b^2+25)b^2$$

$$16b^2 + 400 + 9b^2 = b^4 + 25b^2$$

$$25b^2 + 400 = b^4 + 25b^2$$

$$b^4 = 400$$

$$b^2 = 20$$

Substitute to eqn (ii) , $a^2 = 45$

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8 (c) Therefore equation of an ellipse will be

$$\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$$

8 (d) Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Slope: $\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$2b^2x + 2a^2y\frac{dy}{dx} = 0$$

$$a^2y\frac{dy}{dx} = -b^2x$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

at $P(a\cos\theta, b\sin\theta)$

$$M_T = \frac{-b^2(a\cos\theta)}{a^2(b\sin\theta)}$$

$$M_T = \frac{-b\cos\theta}{a\sin\theta}$$

Equation of the tangent

$$\frac{-b\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$ay\sin\theta + bx\cos\theta = ab(\sin^2\theta + \cos^2\theta)$$

but $\sin^2\theta + \cos^2\theta = 1$

Therefore.

$$ay\sin\theta + bx\cos\theta = ab$$

divide by ab both sides

Equation of the tangent

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$ hence shown.

Also from $M_T = \frac{-b\cos\theta}{a\sin\theta}$

from $M_T M_N = -1$

$$\frac{-b\cos\theta}{a\sin\theta} M_N = -1$$

Slope of the normal $= M_N = \frac{a\sin\theta}{b\cos\theta}$

Equation of normal

$$\frac{a\sin\theta}{b\cos\theta} = \frac{y - b\sin\theta}{x - a\cos\theta}$$

8(d) $by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$$a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - by\cos\theta.$$

$$\sin\theta\cos\theta(a^2-b^2) = ax\sin\theta - by\cos\theta$$

$$a^2-b^2 = \frac{ax}{\cos\theta} - \frac{by}{\sin\theta}$$

Hence $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2-b^2$

Therefore it is not true that equation of the normal at $P(a\cos\theta, b\sin\theta)$ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{ax}{\cos\theta}+\dfrac{by}{\sin\theta}=a^2-b^2$