CHRISTIAN SOCIAL SERVICES COMMISSION (CSSC)
NORTHERN ZONE JOINT EXAMINATIONS SYNDICATE (NZ-JES)

FORM SIX PRE-NATIONAL EXAMINATIONS 2026

142/1 ADVANCED MATHEMATICS 1

MARKING SCHEME

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1. By using a non- programmable calculator:

(a) Evaluate $\sqrt[3]{\left(\frac{\pi^{\tan 30^0}}{3.414}\right)\left(\frac{e^{4.617}+\cos^{-1}(\ln 0.65)}{\sqrt[5]{e^3}}\right)}$, correct to 6 decimal places

(b) Find $\arg\left(\frac{1}{z^4}\right)$, where $z = 3+4i$, correct to three significant figures.

(c) Evaluate $\int_{0.1}^{0.5} \frac{40}{\pi} \cos\left(\frac{\pi t}{2}\right) dt$, correct to four decimal places.

Solution

(a) 3.179057 (04 marks)

(b) $147^0 28'$ (03 marks)

(c) 4.4636 (03 marks)

2. (a) Prove that $\frac{1+\sinh 2x+\cosh 2x}{1-\sinh 2x-\cosh 2x} = -\coth x$

Solution (03 marks)

Consider $L.H.S$

$$= \frac{1+\sinh 2x+\cosh 2x}{1-\sinh 2x-\cosh 2x}$$

$$= \frac{1+2\sinh x\cosh x+\cosh^2 x+\sinh^2 x}{1-\left(2\sinh x\cosh x+\cosh^2 x+\sinh^2 x\right)}$$

$$= \frac{2\sinh x\cosh x+\cosh^2 x+\left(1+\sinh^2 x\right)}{1-\left(2\sinh x\cosh x+\cosh^2 x+\sinh^2 x\right)}$$

$$= \frac{2\sinh x\cosh x+\cosh^2 x+\cosh^2 x}{\cosh^2 x-\sinh^2 x-2\sinh x\cosh x-\cosh^2 x-\sinh^2 x}$$

$$= \frac{2\sinh x\cosh x+2\cosh^2 x}{-2\sinh x\cosh x-2\sinh^2 x}$$

$$= \frac{2\cosh x(\sinh x+\cosh x)}{-2\sinh x(\cosh x+\sinh x)} = -\coth x$$

$L.H.S = R.H.S$

Hence, proved

$= \dfrac{\cosh x}{-\sinh x}$

(b) Given that $\tanh^{-1} u + \tanh^{-1} v = \frac{1}{2}\ln 5$, prove that $v = \dfrac{2-3u}{3-2u}$.

Solution (03 marks)

We have, $\tanh^{-1} u + \tanh^{-1} v = \frac{1}{2}\ln 5$

Required to prove $v = \dfrac{2-3u}{3-2u}$.

Recall: $\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1}\left(\dfrac{x+y}{1+xy}\right)$

$$\tanh^{-1}\left(\frac{u+v}{1+uv}\right) = \ln\sqrt{5}$$

$$\frac{u+v}{1+uv} = \tanh\left(\ln\sqrt{5}\right)$$

$$\frac{u+v}{1+uv} = \frac{e^{\ln\sqrt{5}} - e^{-\ln\sqrt{5}}}{e^{\ln\sqrt{5}} + e^{-\ln\sqrt{5}}}$$

$$\frac{u+v}{1+uv} = \frac{\sqrt{5} - \frac{1}{\sqrt{5}}}{\sqrt{5} + \frac{1}{\sqrt{5}}}$$

$$\frac{u+v}{1+uv} = \frac{5-1}{5+5} = \frac{4}{6} = \frac{2}{3}$$

$$3(u+v) = 2(1+uv)$$

$$3u + 3v = 2 + 2uv$$

$$3v - 2uv = 2 - 3u$$

$$(3-2u)v = 2-3u$$

$$\therefore v = \frac{2-3u}{3-2u} \text{ Proved}$$

(c) If $\sec\theta = \cosh u$, with $u > 0$ and $0 < \theta < \frac{\pi}{2}$, express (i) $\tan\theta$ (ii) $\dfrac{d\theta}{du}$ in terms of u.

Hence, or otherwise, evaluate $\displaystyle\int_0^\infty \frac{1}{\cosh u}\,du$.

Solution (04 marks)

(i) Express $\tan\theta$ in terms of u.

We have, $\sec\theta = \cosh u$

From $1 + \tan^2\theta = \sec^2\theta$

$$\tan\theta = \sqrt{\sec^2\theta - 1}$$

$$\tan\theta = \sqrt{\cosh^2 u - 1} = \sinh u$$

$$\therefore \tan\theta = \sinh u$$

(ii) Required $\frac{d\theta}{du}$ in terms of u.

From (i) above

$\tan\theta = \sinh u$

$d(\tan\theta) = d(\sinh u)$

$\frac{d\theta}{du} = \frac{\cosh u}{\sec^2\theta} = \frac{\cosh u}{\cosh^2 u}$

$\therefore \frac{d\theta}{du} = \operatorname{sech} u$

(iii) $\int_0^\infty \frac{1}{\cosh u} du = \int_0^\infty \operatorname{sech} u\, du$

From (ii) above

$\frac{d\theta}{du} = \operatorname{sech} u \Rightarrow d\theta = \operatorname{sech} u\, du$

$\int_0^\infty \frac{1}{\cosh u} du = \int_0^\infty d\theta = [\theta]_0^\infty$

From $\tan\theta = \sinh u \Rightarrow \theta = \tan^{-1}(\sinh u)$

$= [\tan^{-1}(\sinh u)]_0^\infty$

$= \tan^{-1}(\sinh\infty) - \tan^{-1}(\sinh 0)$

$= \tan^{-1}(\infty) = \frac{\pi}{2}$

$\therefore \int_0^\infty \frac{1}{\cosh u} du = \frac{\pi}{2}$

3. Two Godowns G_1 and G_2 have grain capacity of 100 quintals and 50 quintals respectively. They supply to three ration shops S_1, S_2 and S_3, whose requirements are 60, 50 and 40 quintals respectively. The costs of transportation per quintal from the Godowns to the shops are given in the following table:

Transportation cost per quintal (in tshs)		
Godowns	Shops	Shipping cost per quintal
G_1	S_1	600
G_1	S_2	300
G_2	S_1	400
G_2	S_2	200
G_1	S_3	250
G_2	S_3	300

(i) How the supplies should be transported in order that the transportation cost is minimum

(ii) What is the overall minimum cost?

Solution (10 marks)

Let $x.y$ and $100-x-y$ units should be transported from the plant G_1 to centre S_1, S_2 *and* S_3 Respectively

From \ To	1	2	3	Supply
G_1	x	y	$100-x-y$	100
G_2	$60-x$	$50-y$	$x+y-60$	50
Demand	60	50	40	

Let Z be the total cost

$$Z = 600x + 300y + 250(100-x-y) + 400(60-x) + 200(50-y) + 300(x+y-60)$$
$$= (600-250-400+300)x + (300-250-200+300)y + 41{,}000$$
$$= 250x + 150y + 41{,}000$$

∴ The given LP problem reduces to:

Minimize: $Z = 250x + 150y + 41{,}000$

Subject to the constraints:

$x+y \le 100$ □(1)

$x \le 60$ □(2)

$y \le 50$ □(3)

$x+y \ge 60$ □(4)

$x \ge 0$ □(5)

$y \ge 0$ □(6)

The Graph System

$x+y=100$

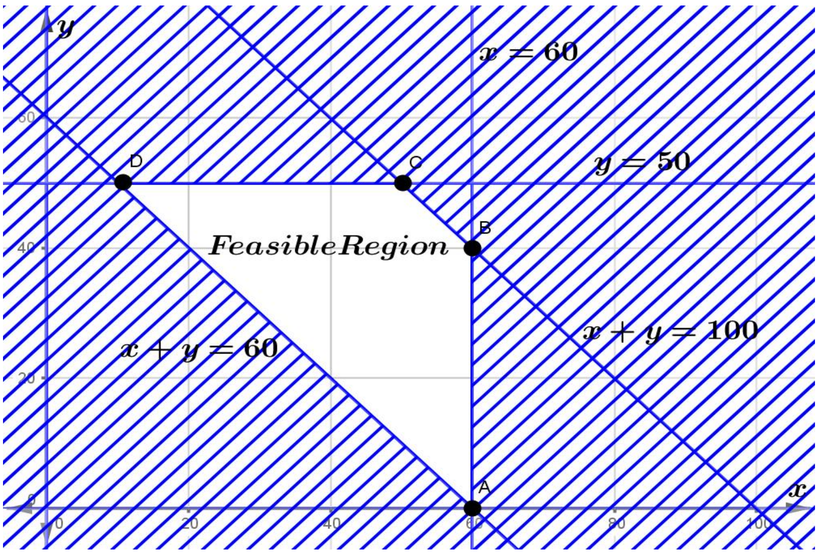
$(0,100)$ *and* $(100,0)$

$x=60$

$y=50$

$x+y=60$

$(0,60)$ *and* $(60,0)$



Corner points	$Z = 250x + 150y + 41,000$
$A(60,0)$	56,000.00
$B(60,40)$	62,000.00
$C(50,50)$	61,000.00
$D(10,50)$	51,000.00

Minimum value of $Z = 51,000.00$ and occurs when $x = 10$ *and* $y = 50$

The units are transported as follows:

To **(units)** / From	S_1	S_2	S_3
G_1	10	50	40
G_2	50	0	0

4. (a) Derive the formula of calculating mean and standard deviation by coding method.

Solution (05 marks)

From $u = (x_i - A)/c$

Where $u =$ Coded data

$x =$ Class mark

$A =$ Assumed mean, $c =$ interval size

$x_i = A + cu_i \quad for\ i = 1,2,3,...n$

So $f_i x_i = f_i A + f_i cu_i$

Therefore,

$$\sum f_i x_i = \sum f_i A + \sum f_i cu_i$$

$$\frac{\sum f_i x_i}{\sum f_i} = A\frac{\sum f_i}{\sum f_i} + c\frac{\sum f_i u}{\sum f_i}$$

$$\bar{x} = A + c\frac{\sum f_i u}{\sum f_i}$$

$$\therefore \bar{x} = A + c\bar{u}$$

Also,

$$s^2 = \frac{\sum f_i\left(x_i - \bar{x}\right)^2}{\sum f_i}$$

$$= \frac{\sum f_i\left[A + cu_i - \left(A + c\bar{u}\right)\right]^2}{\sum f_i}$$

$$= \frac{\sum f_i\left(cu_i - c\bar{u}\right)^2}{\sum f_i}$$

$$= c^2\frac{\sum f_i\left(u_i - \bar{u}\right)^2}{\sum f_i}$$

$$= c^2\left\{\frac{\sum f_i u_i^2}{\sum f_i} - 2\bar{u}\frac{\sum f_i u_i}{\sum f_i} + \bar{u}^2\frac{\sum f_i}{\sum f_i}\right\}$$

$$= c^2\left\{\frac{\sum f u_i^2}{\sum f_i} - 2\bar{u}^2 + \bar{u}^2\right\} \quad \left(\bar{u} = \frac{\sum f_i u_i}{\sum f_i}\right)$$

$$= c^2\left\{\frac{\sum f_i u_i^2}{\sum f_i} - \bar{u}^2\right\} \text{ Where } \sum f_i = N$$

$$= c^2\left\{\frac{\sum f u^2}{N} - \left(\frac{\sum f u}{N}\right)^2\right\}$$

$$S.D = c\sqrt{\left\{\frac{\sum f u^2}{N} - \left(\frac{\sum f u}{N}\right)^2\right\}}$$

(b) The following table gives the frequency distribution of the intelligence quotient X of 500 scholars.

X	130−133	126−129	122−125	118−121	114−117	110−113	106−109
f	6	10	18	28	39	56	75

102−105	98−101	94−97	90−93	86−89	82−85
92	71	49	32	19	5

Find: (i) Standard deviation by coding method using assumed mean =107.5

(ii) Semi-Interdecile range

Solution (05 marks)

(i) We use the code

$u = \frac{x - A}{c}$, $A = 107.5$, $c = 4$

Intelligence quotient	Mid- point x	f	$c.f$	u	fu	fu^2
82−85	83.5	5	5	−6	−30	180
86−89	87.5	19	24	−5	−95	475
90−93	91.5	32	56	−4	−128	512
94−97	95.5	49	105	−3	−147	441
98−101	99.5	71	176	−2	−142	284
102−105	103.5	92	268	−1	−92	92
106−109	107.5	75	343	0	0	0
110−113	111.5	56	399	1	56	56
114−117	115.5	39	438	2	78	156
118−121	119.5	28	466	3	84	252
122−125	123.5	18	484	4	72	288
126−129	127.5	10	494	5	50	250
130−133	131.5	6	500	6	36	216

$\sum f = 500$, $\sum fu = -258$ and $\sum fu^2 = 3202$

From $S.D = c\sqrt{\left\{\frac{\sum fu^2}{N} - \left(\frac{\sum fu}{N}\right)^2\right\}}$

$$S.D = 4\sqrt{\left\{\frac{3202}{500} - \left(\frac{-258}{500}\right)^2\right\}} = 9.91$$

The standard deviation is 9.91

(ii) Required semi decile range

$$S.D.R = \frac{1}{2}(D_9 - D_1)$$

$$D_i = L + \left(\frac{\frac{i}{10}N - f_b}{f_w}\right)c$$

$$D_1 = \left(\frac{N}{10}\right)^{th} = \left(\frac{500}{10}\right)^{th} = 50^{th}$$

D_1 Class $= 90 - 93$

$L = 89.5,\ f_b = 24,\ f_w = 32$ and $c = 4$

$$D_1 = 89.5 + \left(\frac{50-24}{32}\right)4 = 92.75$$

$$D_9 = \left(\frac{9N}{10}\right)^{th} = \left(\frac{4{,}500}{10}\right)^{th} = 450^{th}$$

D_1 Class $= 118 - 121$

$L = 117.5,\ f_b = 438,\ f_w = 28$ and $c = 4$

$$D_9 = 117.5 + \left(\frac{450-438}{28}\right)4 = 119.21$$

$$S.D.R = \frac{1}{2}(119.21 - 92.75) = 13.23$$

5. (a) Using the properties of sets, simplify the following expression:

$$(A \cap B \cap C) \cup (A' \cap B \cap C) \cup B' \cup C'$$

Solution (03 marks)

$(A \cap B \cap C) \cup (A' \cap B \cap C) \cup (B' \cup C')$	Given
$(A \cap A') \cup (B \cap C) \cup (B' \cup C')$	Distributive law
$\mu \cup (B \cap C) \cup (B' \cup C')$	Complement law
$(B \cap C) \cup (B' \cup C')$	Identity law
$(B \cap C) \cup (B \cap C)'$	De Morgan's law
μ	Complement law

$\therefore (A \cap B \cap C) \cup (A' \cap B \cap C) \cup B' \cup C' \equiv \mu$

(b) Describe the following sets by tabular method:

(i) $\{x : x^3 + 1 = 0,\ x \in N\}$

(ii) The set of all letters in the word TRIGONOMETRY.

Solution (03 marks)

(i) $x^3 + 1 = 0 \Rightarrow (x+1)(x^2 - x + 1) = 0$

$x + 1 = 0 \Rightarrow x = -1$, which is not is N

$x^2 - x + 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{-3}}{2}$, which is not is N

$\therefore$ Given set $= \{\ \}$

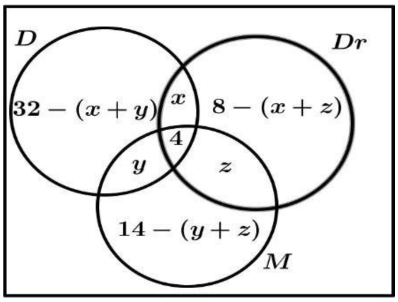
(ii) TRIGONOMETRY are T, R, I, G, O, N, M, E, Y.

$\therefore$ Given set $= \{T, R, I, G, O, N, M, E, Y\}$.

(c) In a competition a school awarded medals in different categories. 36 Medals in Dance, 12 Medals in Dramatics, 18 Medals in Music's. If these Medals went to a total of 45 persons and only four persons who got Medals in all of the three categories. How many received Medal in exactly two of these categories.

Solution (04 marks)

We have, $n(D) = 36, n(Dr) = 12, n(M) = 18, n(\mu) = 45$



n(AuBuC) = n(A)+n(B)+n(C) – n(AnB) – n(AnC) – n(BnC) – n(BnC) – n(AnBnC)

$45 = 32 - x - y + x + 4 + y + x + 4 + z + 8 - x - z + y + 4 + z + 14 - y - z - 4 - y - 4 - z - 4 + 4$

$45 = 58 - x - y - z$

$x + y + z = 13$

$\therefore$The number of Medals received in exactly two of these categories are 13.

6. (a) The function f and g are defined by $f : x \to \frac{x+1}{5}, x \in \Re$ and $g : x \to e^x\ x \in \Re$.

(i) Solve $fog(x) = 17$

(ii) State the range of g.

Solution (02 marks)

(i) Solve $fog(x)=17$

$fog(x)=f[g(x)]=17$

$f(e^x)=17 \Rightarrow \frac{e^x+1}{5}=17$

$e^x=84 \Rightarrow x=\ln 84.$

(ii) State the range of g.

Range of g is $\{y: y\in\Re, y>0\}$

(b) The graph of $y=\frac{ax^2+bx+c}{x^2+qx+r}$ has the lines $y=2, x=1$ and $x=3$ as asymptotes and a turning point at $(0,1)$.

(i) Find the constants a, b, c, q and r.

(ii) Show that the graph has a second turning point.

(iii) Sketch the graph showing clearly its turning point and behaviour as it approaches the asymptote.

(iv) State its domain and range.

Solution (08 marks)

(i) We have,

$$y=\frac{ax^2+bx+c}{x^2+qx+r}$$

Vertical asymptotes are $x=1$ and $x=3$

$\therefore (x-1)(x-3)=x^2-4x+3$

Let $x^2+qx+r=x^2-4x+3$

By comparing the coefficients and constant term

$q=-4,\ r=3$

For $y=2$

$$y=\frac{ax^2+bx+c}{x^2+qx+r}=\frac{a\left(\frac{x^2}{x^2}\right)+b\left(\frac{x}{x^2}\right)+\frac{c}{x^2}}{1+\frac{a}{x}+\frac{r}{x^2}}$$

As $x\to\infty$

$y=a$

Since $y=2$, then $a=2$

At turning point $(0,1)$.

$$1=\frac{a(0)^2+b(0)+c}{(0)^2+q(0)+r} \Rightarrow 1=\frac{c}{r}$$

$c=r$ but $c=3 \Rightarrow r=3$

$$y=\frac{2x^2+bx+3}{x^2-4x+3}$$

Also, required the value of b

$$\frac{dy}{dx}=\frac{\left(x^2-4x+3\right)\frac{d}{dx}\left(2x^2+bx+3\right)-\left(2x^2+bx+3\right)\frac{d}{dx}\left(x^2-4x+3\right)}{\left(x^2-4x+3\right)^2}$$

$$\frac{dy}{dx}=\frac{\left(x^2-4x+3\right)\left(4x+b\right)-\left(2x^2+bx+3\right)\left(2x-4\right)}{\left(x^2-4x+3\right)^2}$$

At turning point $\frac{dy}{dx}=0$, but the turning point is $(0,1)$.

$0=3b+12 \Rightarrow b=-4$

$\therefore a=2, b=-4, c=3, q=-4$ and $r=3$.

The new function is

$$y=\frac{2x^2-4x+3}{x^2-4x+3}$$

(ii) Required the second turning point

From, $\frac{dy}{dx}=\frac{\left(x^2-4x+3\right)\left(4x+b\right)-\left(2x^2+bx+3\right)\left(2x-4\right)}{\left(x^2-4x+3\right)^2}$

$$\frac{dy}{dx}=\frac{\left(x^2-4x+3\right)\left(4x-4\right)-\left(2x^2-4x+3\right)\left(2x-4\right)}{\left(x^2-4x+3\right)^2}$$

At turning point $\frac{dy}{dx}=0$

$0=\left(x^2-4x+3\right)\left(4x-4\right)-\left(2x^2-4x+3\right)\left(2x-4\right)$

$0=4x^3-20x^2+28x-12-4x^3+16x^2-22x+12$

$0=-4x^2+6x$

$x\left(-4x+6\right)=0$

$x=0,\ x=\frac{3}{2}$

From, $y=\frac{2x^2-4x+3}{x^2-4x+3}$, when $x=\frac{3}{2}$

$y=-2$

The second turning point is $\left(\frac{3}{2},-2\right)$

(iii) Its graph and behaviour

Vertical asymptote is the lines $x=1$ and $x=3$

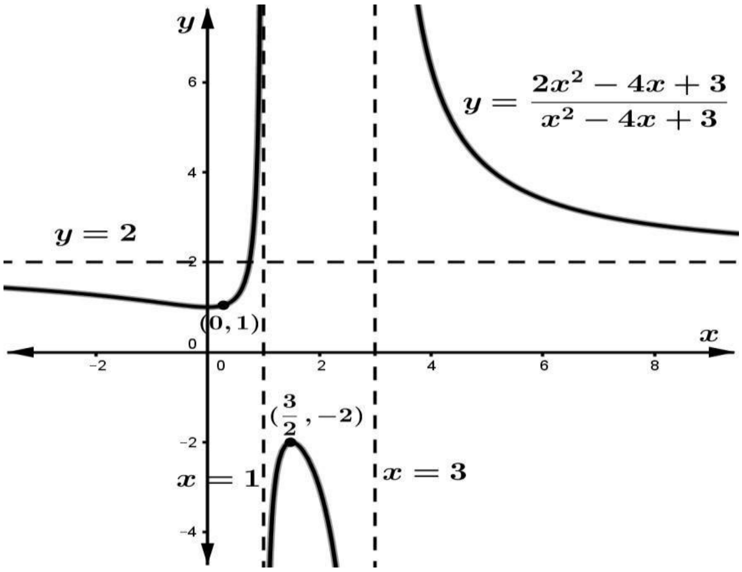
Horizontal asymptote is the line $y=2$

x and $y-$intercepts

If $x=0,\ y=1$

$y=0,$ No $x-$intercept

The turning points are $(0,1)$ and $\left(\frac{3}{2},-2\right)$



The nature of the graph at infinity

As $x \to 1^+$, $y \to -\infty$ and As $x \to 1^-$, $y \to +\infty$

As $x \to 3^+$, $y \to +\infty$ and As $x \to 3^-$, $y \to +\infty$

As $x \to \infty$, $y \to 2$

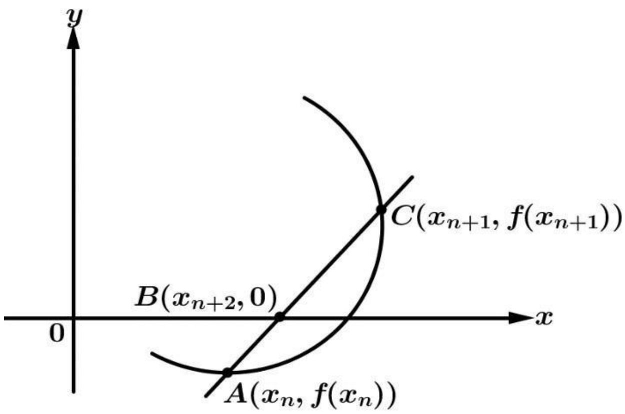
(iv) Domain $= \{x : x \neq 1, x \neq 3\}$

Range $= \{y : y \geq 1 \; and \; y \leq -2\}$

7. (a) (i) Derive the Secant formula for approximating the roots of the function $f(x) = 0$.

Solution (02 marks)

Consider the sketch below



We are to approximate x_{n+2}

From the figure above

Slope $AB =$ slope AC

$$\frac{0 - f(x_n)}{x_{n+2} - x_n} = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

$$\frac{-f(x_n)}{x_{n+2}-x_n}=\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}$$

$$-f(x_n)[x_{n+1}-x_n]=[f(x_{n+1})-f(x_n)][x_{n+2}-x_n]$$

$$\frac{-f(x_n)(x_{n+1}-x_n)}{f(x_{n+1})-f(x_n)}=x_{n+2}-x_n$$

$$\therefore x_{n+2}=x_n-\left[\frac{x_{n+1}-x_n}{f(x_{n+1})-f(x_n)}\right]f(x_n)$$

(ii) Use the formula obtained in 7(a)(i) above to find the solution of $\sin x+xe^x$ between -3 and -4, correct to four decimal places. Perform only three iterations.

Solution (03 marks)

we have, $f(x)=\sin x+xe^x$

$x_0=-3,\ f(x_0)=-0.290481$

$x_1=-4,\ f(x_1)=0.683540$

1[st] iteration $n=0$

$$x_2=x_0-\left[\frac{x_1-x_0}{f(x_1)-f(x_0)}\right]f(x_0)$$

$$x_2=-3-\left[\frac{-4-(-3)}{0.683540-(-0.290481)}\right](-0.290481)$$

$x_2=-3.2982$

$f(x_2)=0.034101$

2[nd] iteration $n=1$

$$x_3=x_1-\left[\frac{x_2-x_1}{f(x_2)-f(x_1)}\right]f(x_1)$$

$$x_3=-4-\left[\frac{-3.2982+4}{0.034101-0.683540}\right](0.683540)$$

$x_3=-3.2613$

$f(x_3)=-0.005612$

3[rd] iteration $n=2$

$$x_4=x_2-\left[\frac{x_3-x_2}{f(x_3)-f(x_2)}\right]f(x_2)$$

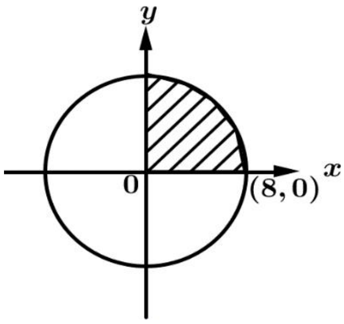
$$x_4=-3.2982-\left[\frac{-3.2613+3.2982}{-0.005612-0.034101}\right](0.034101)$$

$x_4=-3.2665.$

$\therefore$ The root of $\sin x+xe^x$ is -3.2665

(b) Apply both trapezium and Simpson's rule to estimate the area of a quadrant of a circle of radius 8 cm by dividing it into eight intervals. Hence, use the better of these results to find an approximate value of π.

Solution (05 marks)



Eight intervals $\Rightarrow 8$ strips, 9 ordinates

$h = \frac{8-0}{8} = 1$

$x_1 = 0, x_2 = 8$

From general equation of a circle, Centre at origin and radius 8

$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 64$

$y = \sqrt{64 - x^2}$

x	0	1	2	3	4	5	6	7	8
y	8	7.9373	7.7460	7.4162	6.9282	6.2450	5.2915	3.8730	0
ordinates	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By trapezium rule

$A = \frac{1}{2}h[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$

$A = \frac{1}{2}[8 + 2(45.4372)] = 49.4372$ *sq.units*

By Simpson's rule

$A = \frac{1}{3}h[(y_0 + y_8) + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)]$

$A = \frac{1}{3}[8 + 2(19.9657) + 4(25.4715)] = 49.9391$ *sq.units*

To choose the better approximation, we need to know the exact value by using integration

$A = \int_0^8 \sqrt{64 - x^2}\,dx$

$= 8\int_0^8 \sqrt{1 - \left(\frac{x}{8}\right)^2}\,dx$

Let $\frac{x}{8} = \sin\theta \quad \Rightarrow \quad dx = 8\cos\theta d\theta$

$8\int_0^8 \sqrt{1 - \left(\frac{x}{8}\right)^2}\,dx = 8\int \cos\theta \bullet 8\cos\theta d\theta$

$$= 64\int \cos^2\theta d\theta = 64\int \frac{1+\cos 2\theta}{2} d\theta$$

$$= 32\left(\theta + \frac{1}{2}\sin 2\theta\right) + c$$

But $\theta = \sin^{-1}\left(\frac{x}{8}\right)$

$$= 32\left[\sin^{-1}\left(\frac{x}{8}\right) + \frac{1}{2}\sin 2\left(\sin^{-1}\left(\frac{x}{8}\right)\right)\right]_0^8$$

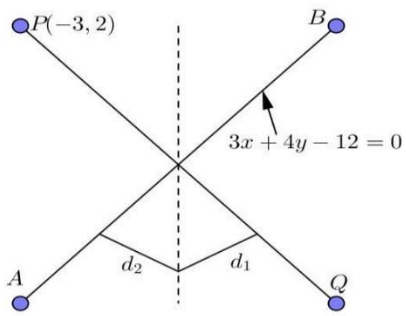
$$= 32\left(\sin^{-1}1\right) = 32\left(\frac{\pi}{2}\right)$$

$$= 16\pi \approx 50.2654$$

Thus the better approximation is Simpson's rule which gives 49.9391 closer to 16π.
Hence, $16\pi \approx 49.9391 \Rightarrow \pi \approx 3.1212$.

8. (a) Q is the foot of the perpendicular from the point $P(-3,2)$ on to a line AB whose equation $3x + 4y = 12$. Find the equations to the bisectors of the angle between the lines PQ and AB.

Solution (04 marks)



$3x + 4y - 12 = 0$

$y = -\frac{3}{4}x + 3$, *let* $m_1 = -\frac{3}{4}$

The equation of the PQ passes through $(-3,2)$

$y - y_0 = m(x - x_0)$

$y - 2 = m(x + 3)$

But it perpendicular to the line AB.

$m_1 m_2 = -1$

$-\frac{3}{4}m_2 = -1 \rightarrow m_2 = \frac{4}{3}$

$y - 2 = \frac{4}{3}(x + 3) \rightarrow 4x - 3y + 18 = 0$

The line PQ is $4x - 3y + 18 = 0$

$$\frac{4x-3y+18}{\sqrt{4^2+(-3)^2}}=\pm\left(\frac{3x+4y-12}{\sqrt{3^2+4^2}}\right)$$

$$\frac{4x-3y+18}{5}=\pm\left(\frac{3x+4y-12}{5}\right)$$

$4x-3y+18=\pm(3x+4y-12)$

For positive,

$x-7y+30=0$

For negative,

$7x+y+6=0.$

(b) A circle with Centre P and radius r touches externally both the circles $x^2+y^2=4$ and $x^2+y^2-6x+8=0$. Prove that the x-coordinate of P is $\frac{1}{3}r+2$, and that P lies on the curve $y^2=8(x-1)(x-2)$.3

Solution **(06 marks)**

We have, $x^2+y^2=4$ and $x^2+y^2-6x+8=0$

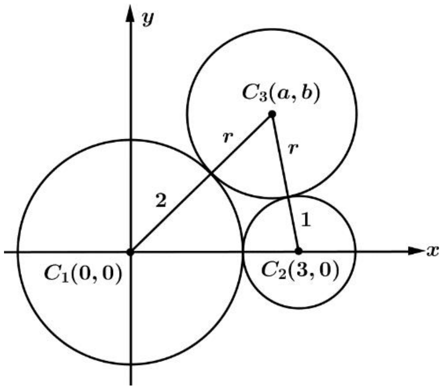
$x^2+y^2=4 \Rightarrow C_1(0,0),\ r=2$

$x^2+y^2-6x+8=0$

$(x-3)^2+(y+0)^2+8-9=0$

$(x-3)^2+(y+0)^2=1 \Rightarrow C_2(3,0),\ r=1$

Let $C_3(a,b)$ be the Centre of the third circle and radius r.



$\sqrt{C_1C_3}=r+2$

$a^2+b^2=(r+2)^2$

$a^2+b^2=r^2+4r+4 \quad \square(1)$

$\sqrt{C_2C_3}=r+1$

$(a-3)^2+b^2=(r+1)^2$

$a^2-6a+9+b^2=r^2+2r+1 \quad \square(2)$

$b^2=r^2+4r+4-a^2$ into (2)

$a^2-6a+9+r^2+4r+4-a^2=r^2+2r+1$

$-6a+9+4r+4=2r+1$

$2r+12=6a$

$a=\frac{1}{6}(2r+12)$

$\therefore a=\frac{r}{3}+2$

Now, to show that P lies on the curve $y^2=8(x-1)(x-2)$

$a=\frac{r}{3}+2$

From, $a^2+b^2=r^2+4r+4$

$b^2=r^2+4r+4-\left(\frac{r}{3}+2\right)^2$

$b^2=r^2+4r-\frac{r^2}{9}-\frac{4}{3}r$

$b^2=\frac{8}{9}r^2+\frac{8}{3}r=\frac{8}{3}r\left(\frac{r}{3}+1\right)$

Let $a=x$, $b=y$

$y^2=\frac{8}{3}r\left(\frac{r}{3}+1\right)=L.H.S$

Consider the $R.H.S$

$8(x-1)(x-2)=8\left(\frac{r}{3}+2-1\right)\left(\frac{r}{3}+2-2\right)$

$=8\left(\frac{r}{3}+1\right)\frac{r}{3}=\frac{8}{3}r\left(\frac{r}{3}+1\right)=R.H.S$

$\therefore L.H.S=R.H.S$

$\therefore$ Point P lies on the curve $y^2=8(x-1)(x-2)$

9. (a) Evaluate the following integral $\int e^{4x}\sqrt{1+e^{2x}}\,dx$.

(**Hint**: Use trigonometric substitution and leave your answer without trig notation)

Solution **(03 marks)**

We have, $\int e^{4x}\sqrt{1+e^{2x}}\,dx$.

$\int e^{4x}\sqrt{1+e^{2x}}\,dx=\int (e^x)^4\sqrt{1+(e^x)^2}$

Let $e^x=\tan\theta \Rightarrow e^x dx=\sec^2\theta d\theta$

$dx=\frac{\sec^2 d\theta}{\tan\theta}$

$=\int \tan^4\theta\sqrt{1+\tan^2\theta}\times\frac{\sec^2\theta d\theta}{\tan\theta}$

$$= \int \tan^3\theta \sec^3\theta\, d\theta$$

$$= \int \tan^2\theta \sec^2\theta \times \sec\theta \tan\theta\, d\theta$$

$$= \int \left(\sec^2\theta - 1\right)\sec^2\theta \times \sec\theta \tan\theta\, d\theta$$

Let $u = \sec\theta \quad \Rightarrow du = \sec\theta \tan\theta\, d\theta$

$$d\theta = \frac{du}{\sec\theta \tan\theta}$$

$$= \int \left(u^2 - 1\right)u^2 du = \int \left(u^4 - u^2\right) du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$$

But $u = \sec\theta \quad \Rightarrow u = \sqrt{1 + \tan^2\theta}$

$$u = \sqrt{1 + e^{2x}}$$

$$= \frac{1}{5}\left(\sqrt{1 + e^{2x}}\right)^5 - \frac{1}{3}\left(\sqrt{1 + e^{2x}}\right)^3 + c$$

(b) Evaluate: $\int \sqrt{\left(\frac{x-2}{5-x}\right)}\, dx.$

Solution **(03 marks)**

We have, $I = \int \sqrt{\left(\frac{x-2}{5-x}\right)}\, dx$

Rationalizing the numerator

$$I = \int \frac{x-2}{\sqrt{(5-x)(x-2)}}\, dx = \int \frac{x-2}{\sqrt{\left(-x^2 + 7x - 10\right)}}\, dx$$

Let $x - 2 = \frac{d}{dx}\left(-x^2 + 7x - 10\right)A + B$

$$x - 2 = \left(-2x + 7\right)A + B$$

$$A = -\frac{1}{2},\ B = \frac{3}{2}$$

$$x - 2 = -\frac{1}{2}\left(-2x + 7\right) + \frac{3}{2}$$

$$I = -\frac{1}{2}\int \frac{\left(-2x + 7\right)}{\sqrt{-x^2 + 7x - 10}}\, dx + \frac{3}{2}\int \frac{dx}{\sqrt{-x^2 + 7x - 10}}$$

$$= -\sqrt{\left(-x^2 + 7x - 10\right)} + c + \frac{3}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}}$$

$$=-\sqrt{\left(-x^2+7x-10\right)}+c_1+\frac{3}{2}\sin^{-1}\left(\frac{x-\frac{7}{2}}{\frac{3}{2}}\right)+c_2$$

$$=-\sqrt{\left(-x^2+7x-10\right)}+\frac{3}{2}\sin^{-1}\left(\frac{2x-7}{3}\right)+c$$

(c) Find the area of the region in the first quadrant enclosed by the x-axis, the line $x=\sqrt{3}y$ and the circle $x^2+y^2=4$.

Solution **(04 marks)**

We have,

$x^2+y^2=4 \quad \square(1)$

$x=\sqrt{3}y \quad \square(2)$

Solve (1) and (2), we get

$\left(\sqrt{3}y\right)^2+y^2=4 \Rightarrow 4y^2=4$

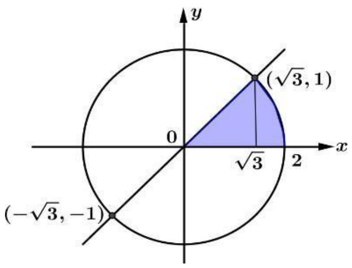
$y^2=1 \Rightarrow y=\pm1$

$y=1 \Rightarrow x=\sqrt{3}(1)=\sqrt{3}$

$y=-1 \Rightarrow x=\sqrt{3}(-1)=-\sqrt{3}$

Two equations intersects at $\left(\sqrt{3},1\right)$ and $\left(-\sqrt{3},-1\right)$

And a circle with Centre at $(0,0)$ and radius 2



$$\text{Area} = \int_0^{\sqrt{3}}\left(y\ of\ line\right)dx+\int_{\sqrt{3}}^{2}\left(y\ of\ circle\right)dx$$

$$=\int_0^{\sqrt{3}}\frac{x}{\sqrt{3}}dx+\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$$

$$=\left[\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}}+\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$=\frac{\sqrt{3}}{2}+\left[2\left(\frac{\pi}{2}\right)-\frac{\sqrt{3}}{2}-2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\pi-2\left(\frac{\pi}{3}\right)=\frac{\pi}{3}\ or\ 1.0472\ sq.units$$

10. (a) If $x=a\sin t-b\cos t$ and $y=a\cos t+b\sin t$, show the $\frac{d^2y}{dx^2}=-\frac{x^2+y^2}{y^3}$

Solution **(3.5 marks)**

We have, $x = a\sin t - b\cos t,\ \ y = a\cos t + b\sin t$

$$\frac{dx}{dt} = a\cos t + b\sin t \qquad (1)$$

$$\frac{dy}{dt} = -a\sin t + b\cos t \qquad (2)$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{-a\sin t + b\cos t}{a\cos t + b\sin t} \qquad (3)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{-a\sin t + b\cos t}{a\cos t + b\sin t}\right)\left(\frac{1}{a\cos t + b\sin t}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(a\cos t + b\sin t)(-a\cos t - b\sin t) - (-a\sin t + b\cos t)(-a\sin t + b\cos t)}{(a\cos t + b\sin t)^3}$$

$$\frac{d^2y}{dx^2} = \frac{-(a\cos t + b\sin t)^2 - \left[-(a\sin t - b\cos t)\times -(a\sin t - b\cos t)\right]}{(a\cos t + b\sin t)^3}$$

$$\frac{d^2y}{dx^2} = \frac{-(a\cos t + b\sin t)^2 - \left[(a\sin t - b\cos t)^2\right]}{(a\cos t + b\sin t)^3}$$

$$\frac{d^2y}{dx^2} = \frac{-(a\cos t + b\sin t)^2 - (a\sin t - b\cos t)^2}{(a\cos t + b\sin t)^3}$$

$$\frac{d^2y}{dx^2} = -\frac{(a\cos t + b\sin t)^2 + (a\sin t - b\cos t)^2}{(a\cos t + b\sin t)^3}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}$$ .Showed

(b) If $z(x+y) = x^2 + y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Solution **(3.5 marks)**

We have, $z(x+y) = x^2 + y^2$, $z = \frac{x^2 + y^2}{x+y}$

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2 + y^2)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y - (x^2 + y^2)}{(x+y)^2} = \frac{-x^2 + 2xy + y^2}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{-x^2 + 2xy + y^2}{(x+y)^2}$$

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=\frac{2x^2-2y^2}{(x+y)^2}=\frac{2(x-y)}{x+y}$$

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\frac{4(x-y)^2}{(x+y)^2} \qquad \square(1)$$

Now,

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$=4\left(\frac{x^2+2xy+y^2-x^2-2xy+y^2+x^2-2xy-y^2}{(x+y)^2}\right)$$

$$=4\left(\frac{x^2-2xy+y^2}{(x+y)^2}\right)=\frac{4(x-y)^2}{(x+y)^2}$$

$$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=\frac{4(x-y)^2}{(x+y)^2} \qquad \square(2)$$

From (1) and (2), we have

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) \text{ Proved}$$

(c) The length x of a rectangle is decreasing at the rate of $5cm/\min ute$ and the width y is increasing at the rate of $4cm/\min ute$. When $x=8cm$ and $y=6cm$. Find the rates of change of (i) the perimeter and (ii) the area of the rectangle.

Solution **(03 marks)**

Since, length x is decreasing

$$\frac{dx}{dt}=-5cm/\min ute$$

The width y is increasing

$$\frac{dy}{dt}=4cm/\min ute$$

(i) The perimeter of a rectangle

$$P=2(x+y)\Rightarrow\frac{dP}{dt}=2\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$$

$$\frac{dP}{dt}=2(-5+4)=-2cm/\min ute.$$

$\therefore$ Perimeter is decreasing at the rate of $2cm/\min$.

(ii) The area A of the rectangle is given by $A=xy$

$$\frac{dA}{dt}=x\frac{dy}{dt}+y\frac{dx}{dt}=(8\times4)+(6)(-5)=2cm^2/\min.$$